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Precautionary saving and asset pricing: implications of separating time and risk preferences

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Precautionary saving and asset pricing:

Implications of separating time and risk preferences

by

Charles Allen Nadeau

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for my wife Elisabet

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CHAPTER 1: INTRODUCTION

It is a topic of active research in consumer and capital theory to determine how to characterize preferences about uncertainty and intertemporal choice. Stochastic choice problems in intertemporal environments have traditionally been approached by assuming that the decision-makers preferences can be represented by a state- and time-separable von Neumann-Morgenstern (VNM) utility function. This representation of preferences offers an elegant and powerful way to analyze temporal behavior under uncertainty.

However, a number of researchers have recently begun to look outside of the expected utility framework to model a number of different intertemporal macroeconomic and microeconomic problems.[1] These efforts have appeared largely in response to the growing body of laboratory evidence which casts doubt upon the descriptive validity of expected utility modeling in general[2] as well as the theoretical and empirical problems encountered by using intertemporal extensions of it. These newer models have also been used in attempts to significantly improve the explanation of non-experimental evidence.

It is well-known that the functional form of a cardinal VNM utility index is constrained in both static and intertemporal environments by several "axioms" which effectively limit the selection of preference representations available to the modeler. In particular, a VNM index is required by a "compound probabilities" axiom to be linear in

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[1] For examples see Hall (1988), Epstein and Zin (1989, 1991), Persson and Svensson (1989), Barsky (1989), Weil (1990), Farmer (1990), Kandel and Stambaugh (1991), Duffie and Epstein (1992), Campbell (1993), and Obstfeld (1994a, 1994b).

[2] For cites see Machina (1982), Duffie (1992), Quiggin (1993), and Harless and Camerer (1994). This laboratory evidence contrasts with a substantial amount of non-experimental evidence that conforms reasonably well with the expected utility hypothesis.

the probabilities in order to generate consistent choice behavior under risk.[3] It is shown in chapter two that this restriction is quite limiting in an intertemporal setting in that the functional forms chosen to model risk attitude and intertemporal substitution cannot be made independently. An important consequence of this functional form inflexibility is that the coefficient of relative risk aversion and the elasticity of intertemporal substitution are artificially constrained to be reciprocals, and thus indistinguishable, in the standard isoelastic and time-additive case.[4]

The issue addressed in this paper is fundamentally a modeling one and involves a characterization of intertemporal preferences under uncertainty that produces consistent choice behavior for an individual, and which also relaxes the preference restrictions implicitly imposed by maintaining all of the axioms of the expected utility hypothesis.[5] A "generalized" expected utility (GEU) model of individual preferences is used which relaxes the preference restrictions by a slight weakening of the expected utility axioms, and which generates choice behavior that generally is not replicated by using a single cardinal utility function over outcomes.

This weakening of the axiomatic structure underlying expected utility modeling occurs by abandoning the axiom relating to the treatment of compound probabilities in a way first suggested by Selden (1978). The generalized model used here maintains all of the other axioms of expected utility and is used in an attempt to clearly understand the intertemporal factor allocations of individuals facing income risk. In addition, a recursive

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[3] This linearity requirement refers to a model with a discrete number of possible states. In a model with an infinite number of states, it can be reinterpreted as linearity in the expectations operator. Thus, by discarding this axiom a modeler produces a utility index that is generally nonlinear in the probabilities or expectations operator.

[4] The elasticity of intertemporal substitution is an important preference concept in economics; see Hall (1988) for discussion. It can be thought of as measuring the change in consumption growth precipitated by a change in the intertemporal price of consumption, where the price is usually taken to be the current real interest rate. The model used in chapter three demonstrates that when leisure as well as consumption enters the utility function the intertemporal price of leisure (i.e. the intertemporal wage ratio) also affects the willingness to transfer resources across time periods.

[5] As Farmer (1990) points out in a multiperiod context, it is also difficult to derive closed-form solutions to stochastic intertemporal choice problems when all of the axioms are retained since such problems quickly become intractable.

infinite-horizon extension of the basic two period framework is evaluated for purposes of pricing assets.

Two questions are explicitly addressed in the paper: (i) does independent parameterization of risk aversion from intertemporal preference significantly aid in the theoretical understanding of the precautionary saving and labor supply behavior of individuals under income risk?, and (ii) does independent parameterization of risk aversion from intertemporal preference for a representative agent significantly aid in the empirical explanation of the high volatility of real returns observed in U.S. asset markets?

The organization of the paper is as follows. Chapter two will present a theoretical discussion of intertemporal expected utility as well as the restrictions that are implicit in the construction of a single cardinal utility index. It shows how a fundamental confusion of risk attitude and intertemporal preferences occurs which, as already noted, produces a precise mathematical constraint on the relation between two distinct preference concepts.[6] This confusion occurs because the expected utility hypothesis artificially constrains the range of functional forms that may be used to characterize these preferences in order to produce a utility index that is (conveniently) linear in the probabilities. As a result, the interpretation of comparative static results is ambiguous since, as Lucas (1978, p. 1441) realized, a change in the concavity of a VNM cardinal utility index simultaneously affects both risk attitude and the degree of intertemporal substitution.[7]

An attempt is made to justify the use of the generalized representation of preferences appearing in chapter three which relaxes these restrictions on functional form and thus includes the expected utility framework as a special case. Work by Selden (1978) and Kreps and Porteus (1978) provides the theoretical support needed to assert that the

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[6] The preferences are obviously distinct since they each determine the concavity of different utility surfaces. Obstfeld (1994a, pp. 1476-1477) clarifies this by observing that intertemporal preference is a relevant preference concept in the absence of uncertainty, while risk attitude is not.

[7] Footnote 8 in Lucas' paper states, "In [a] multiperiod environment, the term 'risk aversion' is perhaps misleading, since the curvature of [the VNM utility index] also governs the intertemporability of consumption. With time-additive utility, there is *no way* [my italics] to disentangle these conceptually distinct aspects of preferences." The model of chapter three demonstrates that his second point applies only within the framework of expected utility.

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disposal of the compound probabilities axiom does not necessarily imply consistency problems in observed choice behavior, which is important since the issue of preference consistency is not trivial when attempting to work outside the expected utility framework. It is argued that the generalized model is able to disentangle preferences at a reasonable "cost" in terms of the necessary deviation from the axiomatic structure of expected utility theory.

It is also argued that perhaps the greatest benefit of modeling with GEU preferences is that by relaxing the compound probabilities axiom, complete freedom in specifying the functional forms representing risk attitude and time preference results - a freedom which does not exist if the axiom is retained by using an expected utility representation of preferences.

A simple two-period model of individual behavior under income risk appears in chapter three which allows independent specification of the functional forms representing risk attitude and intertemporal preference. The model departs from expected utility but remains additively-separable over time, with one concave function describing risk aversion in period two while a second utility function describes intertemporal substitutability between first and second period indirect utility.[8]

In the model, the individual faces three possible sources of real second period income risk and must make decisions in period one regarding the allocation of factor supplies over both periods, where the factor supplies are capital and labor. Primitive preferences are defined over leisure and the consumption of a single good. The indirect utility functions for period one and two are derived from the primitive utility function and used as the arguments in the intertemporal utility optimization problem. The three real income sources considered in the model are: (i) pure endowment income, (ii) capital return

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[8] One drawback of the nonrecursive representation of preference in chapter three is that it does not generalize beyond two periods (see Hall (1988)). However, recursive extensions of it are possible, as shown in chapter five.

income, and (iii) labor income.[9] Cases are considered in which only one of the income sources is stochastic as well as cases in which two income sources are simultaneously stochastic. The model is partial equilibrium in nature as no explicit attempt is made to identify the underlying cause of the income uncertainty in the second period.

This work attempts to determine the motivation, at a microeconomic level, behind the intertemporal resource allocation decisions of individuals who confront income uncertainty. It is shown that an individual may have incentives to use precautionary saving as a form of ex ante insurance against future income risk. Similarly, an individual may also have incentives to use future labor supply as a form of ex post insurance. The use of factor supplies as a way to hedge against risk is examined explicitly in the chapter.

We will see that a consequence of using different utility functions to represent intertemporal preference and risk attitude is that the coefficient of absolute risk aversion appears as a separate parameter from the elasticity of intertemporal substitution, something which is not possible under the expected utility theory.[10] This separation of the two critical preference parameters is used to demonstrate the fundamentally asymmetric roles they play in motivating intertemporal resource allocations, particularly saving behavior.

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[9] In general any earnings which involve uninsurable and nontradable risk due to the absence of complete insurance and asset markets will affect current behavior for a risk-averse individual. Examples of real endowment income risk in the U.S. may include uncertainty regarding: government tax and expenditure policies (e.g. budget deficits, taxation of government transfer income such as social security, business and farm subsidies, Medicare and Medicaid, AFDC, food stamps, unemployment benefits, and other income support and social insurance programs), government business and environmental regulation, monetary policy and inflation, and uninsured medical expenses. Examples of real wage income risk may occur due to uncertainty regarding: government taxation of wage income, minimum wage legislation, monetary policy and inflation, and government budget deficits. Examples of real capital return risk may include uncertainty regarding: government taxation of capital income (e.g. from retirement saving accounts, insurable annuities, mortgage interest payments, stock and bond holdings, ect...), government budget deficits, monetary policy and inflation, government restrictions on domestic and international trade in assets and commodities, and government business and environmental regulation.

[10] To derive closed-form analytic decision rules it often proves convenient in intertemporal settings to assume constant elastic utility forms, particularly with expected utility. However, to highlight the limitations of using an intertemporal expected utility model, the utility function characterizing attitude toward risk in chapter three is assumed to be negative exponential instead of isoelastic; either preference assumption would yield closed-form solutions in the model. Intertemporal preferences, however, are assumed to have constant elasticity in deterministic environments.

We will also see that the assumptions of homothetic intertemporal preference and perfect capital markets, in the sense that only one interest rate exists for borrowing and lending purposes in the model, are crucial to obtaining the asymmetric results of chapter three.[11]

By moving from the univariate cases to the multivariate cases it will be shown that intertemporal factor allocations depend not only on the variability of relative factor returns but also on any correlation that may exist between them as well - this correlation can magnify, dampen, or even reverse the comparative static implications in the univariate cases.

In order to demonstrate the importance of the results in chapter three, an isoelastic expected utility model that is typical of those appearing in the received literature is used in chapter four to generate results that are directly comparable to the results in chapter three. The treatment of the two key preference parameters in the two models is contrasted, and classical results from the theoretical literature are shown to hold only because of the way that preferences are entangled in an intertemporal expected utility framework.

The asymmetric treatment of risk attitude and intertemporal preference in the generalized model of chapter three are shown not to hold in chapter four for a trivial reason: a single parameter represents both types of preference in the expected utility model. Thus the model treats them symmetrically in terms of motivating factor allocations in the face of income risk by making them indistinguishable from one another. It is argued that this confusion of tastes inherent to expected utility has resulted in the elevation of risk aversion and the neglect of intertemporal preference considerations in the classic literature.[12]

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[11] Relaxing either of the assumptions would prove an interesting extension of the results in chapter three, but is not undertaken in this paper.

[12] e.g. Arrow (1964) labors on the issue of risk aversion and the viability of competitive allocations of risk in asset markets without addressing the role of intertemporal preferences - Lucas (1978) does likewise with regard to asset market efficiency. Additional references follow in chapter two.

A different application of GEU modeling is explored in chapter five, which examines the usefulness of a recursive infinite-horizon class of preferences introduced by Epstein and Zin (1989, 1991) for asset pricing purposes. The preference representation uses a Kreps-Porteus (1978, 1979a, 1979b) utility index that is constructed recursively (and thus is not linear in the probabilities) and demonstrates that the compound probability axiom is not necessary to guarantee time-consistent preferences in a dynamic setting.[13] This particular class of preferences has been used in a number of recent theoretical and empirical studies.[14]

Standard consumption-based asset pricing models, which use expected utility to model the preferences of a representative agent, and which attempt to explain the temporal aspects of asset returns using consumption growth patterns over time, encounter a troublesome fact: asset returns are historically a high-frequency time series, whereas aggregate and per-capita consumption growth is historically a low-frequency time series in most countries, including the U.S. As a result, attempts to match moments have proven extremely difficult unless questionable preference assumptions are made in the model.[15]

However, it is shown that the recursive model of Epstein and Zin suggests that its use may overcome such difficulties. In particular, from the construction of these preferences it appears that moderate levels of risk aversion coupled with high tolerance for intertemporal substitution [16] will generate substantial amounts of aggregate risk via the consumption growth process, and thus may be used to explain high asset return volatility more convincingly than conventional models based on expected utility.

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[13] Note that recursive preferences are defined in nonstochastic environments by the assumption of time-separability (Kreps and Porteus (1978)), but not in a stochastic environment - thus, unlike expected utility, the Epstein-Zin class of preferences are generally time-nonseparable.

[14] For examples see Prasad (1991), Kandel and Stambaugh (1991), Campbell (1993), Obstfeld (1994a, 1994b), Hung (1994), and Epstein and Melino (1995). The model has been extensively examined in Altug and Labadie (1992); Duffie and Epstein (1992) use a continuous-time version of the model which, unlike the discrete-time version, is immune from the Roll (1977) critique of the CAPM.

[15] These difficulties are related to the well-known "equity premium" and "risk-free rate" puzzles that appear in the finance literature and that are addressed in chapter five.

[16] Note that this preference mixture is not possible when using isoelastic expected utility since then the two preference parameters are constrained to be reciprocals.

An interesting result derived from this model is the emergence of both consumption growth and market return as variables driving asset return over time, so that the Epstein-Zin model is essentially a linear two-factor model that combines the basic results of both the static CAPM and the intertemporal C-CAPM.

In addition, a useful feature of the model is that the recursive preference representation is valid regardless of the information set of the representative agent (Kocherlakota, 1995), unlike virtually all asset pricing models based on nonrecursive preference structures.

Despite its advantages, however, the basic result of chapter five is that the confusion of preferences that occurs within the expected utility framework apparently does not account for the poor empirical performance of the static CAPM or intertemporal C-CAPM, in that the ability of the Epstein-Zin class of preferences to separate time and risk preferences is shown to be unhelpful in explaining the high variability of historic U.S. asset return data when using standard functional forms and data sets.

CHAPTER 2: EU AND GEU PREFERENCES

Introduction

Within the economic theory of individual choice under risk, an old question exists: why do optimizing individuals choose to allocate resources across time when uncertainty is confronted?[1]

Two complementary perspectives in the traditional literature which address the issue of intertemporal resource allocation are the life-cycle hypothesis, which views these allocations as resulting from individual desire to provide for income in old age, and the permanent income hypothesis, which views them as resulting primarily from income-smoothing preferences.

Another way of characterizing these perspectives is that individuals transfer resources across time during different periods of life because of both a life-cycle motive in view of anticipated declines in future income and a precautionary motive which arises because people are in general risk averse and hedge against unanticipated declines in future income (or unanticipated rises in future expenditures).

This study will concern itself with the second motive. Traditional analysis on precautionary intertemporal resource transfers has relied almost exclusively on intertemporal extensions of the classic expected utility model of choice under uncertainty. However, relatively recent theoretical work has discovered that these models impose additional restrictions on preferences beyond the usual ones encountered in the static framework. In addition, closed-form solutions to stochastic intertemporal choice problems are usually difficult to obtain when all of the axioms of the expected utility hypothesis are maintained, since such problems quickly become intractable.

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[1] Throughout the text the terms "uncertainty" and "risk" will be used interchangeably, where both are defined here as indicating randomness with a subjectively-perceived probability distribution.

It is well known that independent specification of the functional forms representing individual time and risk preference is not possible within the expected utility framework, since it uses a single cardinal utility index to capture both preference concepts.[2] It is less well known that this functional form inflexibility causes the commonly used class of constant elasticity of substitution (CES) and time-additive expected utility models to constrain the coefficient of relative risk aversion (CRRA) to be the reciprocal of the elasticity of intertemporal substitution (EIS), which is a strong quantitative restriction on preferences.[3] Moreover, this mathematical restriction represents an unintended axiomatic constraint on distinct preference concepts which has no behavioral or economic rationale, and makes it impossible to model the behavior of individuals who, as empirical work suggests, are extremely averse to intertemporal substitution but who display only a moderate level of risk aversion (Weil, 1990).

In the context of this study, the constraint also makes it impossible, at a microeconomic level, to disentangle the two preference concepts to determine which is responsible for motivating individual consumption and leisure demand under income risk, and motivates the effort to assess, at a macroeconomic level, whether this confusion of preferences significantly contributes to the poor empirical performance of asset pricing models which incorporate a representative agent with expected utility preferences.

More generally, a serious consequence of this functional form inflexibility is that the effects of increased risk aversion on behavior cannot be determined using expected utility. As Lucas (1978, p. 1441) realized in his attempt to understand the determinants of equilibrium asset prices, a comparative static analysis based on the curvature of an expected utility index does not allow an unambiguous interpretation since both risk attitude and intertemporal preferences are changed in this way.

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[2] To my knowledge this observation was first made in Pollak (1967) and was explicitly addressed in Selden (1978) and Rossman and Selden (1978).

[3] Selden (1978, 1979)) and Epstein and Zin (1989) address this issue exhaustively, and it is also commented on by Hall (1988) and Weil (1990). The earliest mention of this constraint on preferences that I have been able to find is in Dreze and Modigliani (1972, pp. 314-315).

It is possible to imagine many other theoretical and empirical areas in economics (e.g. understanding the precautionary saving motive) in which the effects of a change in risk attitude may be of interest.

As a result of the recent and growing dissatisfaction with the theoretical and empirical predictions of models with time- and state-separable expected utility preferences, alternative representations of preferences have developed that move away from the expected utility approach in an attempt to derive advantages for both theoretical and applied work in macroeconomics, finance, and game theory.[4] Many results in diverse areas of economic analysis have been shown to be robust to these generalizations of expected utility.

These new intertemporal models generalize time-additive expected utility, which is standard in capital theory, by relaxing one or more of the fundamental axioms underlying the expected utility hypothesis, and thus represent a more general way of characterizing preferences.[5] A functional form flexibility is achieved that can be exploited by the modeler to analyze the effects on observed behavior of preference assumptions that are otherwise impossible to model using expected utility.

As indicated above, a primary motivation behind this effort is the desire to disentangle intertemporal substitution from risk aversion. The separation of preferences can occur by abandoning the expected utility axiom relating to compound lotteries, which then permits the explicit modeling of risk aversion and intertemporal preference with independently specified functional forms. As the results in chapters three and five demonstrate, the added analytical power produces new theoretical insights and new, testable implications regarding market behavior.

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[4] The cites in this chapter include work in asset pricing, insurance and lotteries, stock market investment behavior, and the theory of social choice as examples.

[5] Standard EU modeling has shown that risk aversion alone is insufficient to predict the precautionary saving behavior of individuals who are faced with income uncertainty (see Leland, 1968, and Sandmo, 1970). An important goal of the newer work is to explicitly model this insight in a more intuitively satisfying way than what is possible by adhering to all of the expected utility axioms (Weil, 1990).

A discussion of the intertemporal expected utility (EU) representation of preferences, as well as the simple two period "generalized" expected utility (GEU) model used in chapter three to better understand how uninsured earnings can alter optimal precautionary saving and labor supply behavior[6], will follow along with commentary on the basic interrelationship between EU and GEU modeling. A general literature review of this material is then presented. Finally, an introduction to the use in chapter five of a recursive extension of the two period GEU model for analyzing asset market data and understanding investment behavior will conclude the chapter.[7]

Two Period EU Preferences

In chapter four of this study a temporal utility index U is defined over two periods with a continuous probability distribution $f(x_2)$ for the period two random variable x_2 (e.g. indirect utility) of the form

$$U(x_1, f(x_2)) = u(x_1) + \int u(x_2) f(x_2) dx_2$$

This preference representation is, when subject to a set of specific axiomatic restrictions (see Fishburn, 1982, for details), called a continuous-state two period cardinal "von Neumann-Morgenstern" (VNM) utility index, after the pioneering work on decision theory by von Neumann and Morgenstern (1944).[8] If preferences were not assumed stationary, then the VNM index would be expressed as

$$U(x_1, f(x_2)) = u_1(x_1) + \int u_2(x_2) f(x_2) dx_2$$

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[6] The basic model framework that is used in chapter three is drawn from Werner (1990) in a slightly modified and simpler form. The main extension involves the consideration of multiple income shocks and single income shocks other than what Werner considers, as well as the use of a more robust technique for dealing with uncertainty in the models.

[7] A literature review for this work is presented separately in chapter five.

[8] Note that this index is a function of first period activity as well as a probability function, unlike a static index.

where u_1 and u_2 are different functions.[9]

The theory of VNM (also called "expected utility" (EU)) characterizes uncertain prospects as probability densities (or just probabilities in the discrete-state case) over a set of outcomes. Thus, probability densities are given as part of the description of an object and are therefore objective. Preferences have an EU representation where each possible outcome (an infinite number in the continuous case) has a corresponding utility level, and the value of a first period outcome and second period probability distribution is measured by the expected level of utility it provides.

An important foundation underlying expected utility theory is the "reduction of compound lotteries" axiom. In a simple static two-state environment, this axiom requires that for all probabilities q and p, and for all outcomes x and y, it must be true for the individual that $(q \circ (p \circ x \oplus (1-p) \circ y) \oplus (1-q) \circ y) \sim (qp) \circ x \oplus (1-qp) \circ y$, which may be interpreted as follows: a lottery (or "gamble") which offers to pay outcome y with probability 1-q or the chance to play a second lottery with probability q, where the second lottery pays outcome x with probability p or outcome y with probability 1-p, must be viewed indifferently to a lottery which pays outcome x with probability qp or outcome y with probability 1-qp (Varian, 1992).

This "reduction" axiom, which allows for a particularly simple representation of otherwise complex choice opportunities, is a necessary (but not sufficient) restriction of the preference representation to ensure that the utility of any event involving uncertain outcomes can be expressed as the expectation of the utility from each possible outcome, in that for the binary case above

$$U(p \circ x \oplus (1-p) \circ y) = pu(x) + (1-p)u(y)$$

This representation of preferences exhibits what is called the "expected utility property", in that the utility index U is constrained to be both additive across outcomes or "states" and

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[9] Notice that the functions U and u are not the same here. U is defined over probabilities and is the VNM index, whereas u is defined over outcomes and can assume a variety of different forms. The specific structure of the U index is very restricted under the expected utility hypothesis; in particular, the index must be time-separable and linear in the probabilities to be a VNM index.

linear in the probability terms, which are particularly convenient assumptions to make when attempting to analyze choice behavior under uncertainty.[10,11]

For several years there has been a debate in the theoretical literature as to whether the preferences of a decision-maker should be artificially restricted this way in an intertemporal framework. The debate has focused on (i) the implicit and subtle restrictions on intertemporal preferences that the reduction axiom imposes, which many researchers view as too rigid, and (ii) the relative merits of abandoning the substitution axiom for the purposes of stochastic intertemporal modeling, which will be the focus of chapters 3 and 4 in the context of the consumption-saving decision under income risk and chapter 5 in the context of asset pricing.

A primary problem of adopting the substitution axiom is that although risk aversion and aversion to intertemporal substitution are two conceptually distinct characteristics of an individual's tastes, they cannot be independently parameterized in the time-separable expected utility framework[12], which in the popular time-additive isoelastic case constrain the coefficient of relative risk aversion (CRRA), which determines the curvature of the within-period utility index, to be the reciprocal of the elasticity of intertemporal substitution (EIS), which determines the curvature of the between-period utility index.[13, 14]

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[10] Three additional axioms are necessary in order to represent preferences this way - see Fishburn (1982). The requirement is modified to read "linear in the expectations operator" in the infinite-state case of chapter four. As Kreps and Porteus (1978) show, the reduction axiom also implies that an individual with preferences conforming to this axiom must be indifferent to the timing of the resolution of uncertainty in a multiperiod environment.

[11] This axiom produces intertemporal preference restrictions that will be the focus of our attention, and motivates the use of an ordinal certainty equivalent (OCE) representation of preferences in chapter three. In such an intertemporal setting it is often assumed, although not required by the axioms of expected utility theory, that the index U is also additive over time periods. There is no compelling reason why a representation of preferences has to be additive for both intertemporal and uncertain choices, but such an assumption makes for easier calculation.

[12] Nor can they be cleanly separated in equally popular time-nonseparable representations, such as in habit persistence and local substitution models (see Duffie and Epstein, 1992, and Epstein and Melino, 1995). Note that this problem is not restricted to just time-additive expected utility models - a confusion of preferences will occur in any intertemporal expected utility model due to the "substitution" axiom discussed later in this section.

[13] In the context of maximizing intertemporal indirect utility as in chapter 3, the EIS is a measure of the response of the rate of change of indirect utility to changes in the expected intertemporal price of resource

Furthermore, this restriction does not represent an explicit, intentional modeling of individual preferences but is rather an implicit mathematical byproduct of adopting the substitution axiom underlying the expected utility hypothesis, which does not allow for independent specification of the functional forms used to represent intertemporal preference and risk attitude.

As an interesting example of this problem in the received literature, consider a standard function with the infinite-horizon form

$$U(\tilde{v}) = E_0 \sum_0^\infty \beta^t u(\tilde{v}_t)$$

where $\tilde{v}_t$ denotes random indirect utility at time t, $0 < \beta < 1$ is the time discount factor, and E_0 is an expectations operator conditional upon period 0 information. A great deal of standard capital theory assumes that the ranking of intertemporal stochastic programs (typically defined over consumption levels rather than indirect utility) can be represented in this time-additive way.

Of particular interest is the curvature of the utility function u which is measured by $-vu''(v)/u'(v)$. This elasticity is the measure of relative risk aversion with respect to indirect utility gambles in any single period. It is also inversely related to the willingness to substitute indirect utility across time. For example, in the case of the common homogeneous specification

$$u(v) = \frac{v^{\delta_1}}{\delta_1} \quad \text{for } \delta_1 \neq 0$$

$$u(v) = \log(v) \quad \text{for } \delta_1 = 0$$

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transfers, where the intertemporal price is a function of both the expected real interest rate and the expected relative wage in period 2. Thus, a lower (higher) EIS represents a lower (higher) sensitivity to relative factor price changes and a higher (lower) propensity to engage in "utility-smoothing" activities in an attempt to maximize intertemporal welfare.

[14] Weil (1993) notes that the EIS is an important preference parameter with respect to the propensity to consume and precautionary saving. A graphical interpretation would be that the desired income time profile becomes flatter, ceteris paribus, when individuals become more averse to intertemporal substitution (i.e. when the EIS parameter falls).

which delivers closed form solutions, the constant elasticity of intertemporal substitution (EIS) is $(1-\delta_1)^{-1}$ while the coefficient of relative risk aversion is $(1-\delta_1)$. Thus, a precise reciprocal relation is imposed on these preferences a priori.[15]

This restriction is unfortunate, since the two preferences are conceptually distinct: risk aversion, as represented by either the absolute or relative measure[16], concerns attitude toward the variation of indirect utility across states of the world (at a given time), while the elasticity measure of intertemporal preference concerns attitude toward the variation of indirect utility across time (in the absence of uncertainty).[17]

Since pure risk attitude is defined only over uncertain prospects, whereas pure "time" preference, as measured by the EIS, is defined only over certain prospects, these preferences are clearly distinct, although not necessarily independent from a behavioral standpoint.[18]

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[15] This result highlights a further problem - risk neutrality necessarily implies that current and future indirect utility will be perfect substitutes to the individual, so that the intertemporal indifference surface will be flat and corner solutions will result in an optimization. There is certainly no economic or behavioral reason to believe that risk neutrality must imply the complete absence of a desire for income smoothing. Worse still, risk loving behavior in an isoelastic expected utility model implies a concave intertemporal indifference surface!

[16] The absolute risk aversion measure over an uncertain outcome x is defined as: $-u''(x)/u'(x)$, where the primes denote derivatives, while the relative risk aversion measure is defined as: $-xu''(x)/u'(x)$. A "constant" relative risk aversion measure is customarily assumed in the literature in order to derive closed-form analytical solutions.

[17] The EIS measure is calculated in a two period context as: $(u'(x_1)u'(x_2)[u'(x_1)x_1 + u'(x_2)x_2])/(x_1x_2D)$ where D is a complex linear equation of first, second, and cross-partials of the utility index with respect to first period activity x_1 and second period activity x_2. The measure determines the degree to which first and second period activities complement one another and "go together" to jointly enhance utility in the mind of the individual.

[18] It is clear from the literature and with discussions with colleagues that this view is not universal; however, from a purely mathematical and modeling standpoint there can be no doubt that these parameters play different roles. In addition, the claim that time and risk preferences are related is not at issue - isoelastic EU models require that a precise mathematical relation exists between these preference concepts that goes beyond simple correlation - these models infer that the relations are, in fact, one and the same thing! Furthermore, a conjecture that the pairings (low substitution, high risk aversion) and (high substitution, low risk aversion) are more empirically plausible than other possibilities since one either strongly dislikes "change" or does not can never be tested unless the constraint on preferences is first broken.

The temptation is great to simply apply a concave transform to the original EU objective function in an attempt to separate time and atemporal risk preferences such that the utility index is

$$U^0(\tilde{v}) = E_0 \, T \left(\sum_0^\infty \beta^{\,t} \, u(\tilde{v}_t) \right)$$

where T is an increasing, concave function.[19] However, a problem emerges here: the transform is still defined over uncertain indirect utility levels, so that the curvature of T will reflect both time and risk preferences since the willingness to substitute current with future utility will depend on risk attitudes as well as intertemporal preferences - this problem will remain as long as any uncertainty remains within the utility index, as Lucas (1978) recognized.

As Epstein and Zin (1989, pp. 951-952) point out, another unappealing feature of a transformed EU objective function is that preference orderings will generally depend on past consumption values in an implausible way if tastes are assumed stationary.[20,21] They also demonstrate that very restrictive assumptions are needed to ensure the stationarity of preferences.

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[19] This attempt represents a translation of the Kihlstrom and Mirman (1974) model of multicommodity risk aversion to a temporal setting. Note that U^0 retains the expected utility property and is ordinally equivalent to U.

[20] The term "implausible" is used since risk attitude will depend more heavily on activity occurring in the distant past than on more recent activity if U^0 is used, in that the risk premium for a small gamble in $\tilde{v}_t$ will be affected more by a small change in v_0 than by a corresponding change in v_{t-1}. This preference anomaly is observed even with many VNM indices that are not additively-separable over time (see Epstein and Zin). A possible exception to this result is the use of an exponential transform - see van der Ploeg (1993), who uses this approach to distinguish between time and temporal risk preferences in an EU model.

[21] If the indirect utility program $\tilde{v} = (\tilde{v}_0, \tilde{v}_1, ..., \tilde{v}_t, \tilde{v}_{t+1}, ...)$ is preferred to the alternative program $\tilde{v}' = (\tilde{v}_0, \tilde{v}_1, ..., \tilde{v}'_t, \tilde{v}'_{t+1}, ...)$ at time 0, then preferences are "stationary" over time if the same is true at time t, where the indirect utility levels between time 0 and time t-1 are assumed identical between the two programs. This assumption ensures the dynamic consistency of preferences, in that preferences can vary through time only because the indirect utility history does - a natural and common assumption in the literature. The absence of it implies that at each time t the individual acts as though time begins anew by totally disregarding the past and using the original utility function to evaluate the future - such an assumption has produced limited results in the literature (Epstein and Zin).

Thus, concave transforms of either two period or multiperiod EU utility indexes do not represent an attractive way to resolve the problem of how to distinguish between intertemporal preferences and risk attitude in the intertemporal utility index.

As van der Ploeg (1992) notes, other problems with using intertemporal EU models include assuming very specific discrete and continuous probability distributions for future variables in order to derive solutions and relying heavily on numerical, as opposed to closed-form analytical, results except in rare instances.[22]

The reduction axiom has been identified as imposing undesirable constraints on behavior and was briefly discussed earlier. This axiom of the expected utility hypothesis states that an individual only cares about the final probabilities of obtaining various outcomes of a gamble and not how the probabilities are formed. It would be violated if, say, the individual has a preference for suspense, and there is some empirical evidence that people treat "compound" lotteries differently than "one-shot" lotteries (Kreps and Porteus, 1978).

Another problem with the reduction axiom, as noted above, is that it implies an indifference as to the temporal resolution of uncertainty. Intuition and introspection suggest that one would rather know today what income next year will be rather than have to wait until the beginning of next year to find out, if for no other reason than planning this year's budget. However, the reduction axiom dictates that an individual should be indifferent as to when she obtains this information due to the offsetting influences of risk aversion and aversion to intertemporal substitution on timing preferences (Kreps and Porteus, 1978, 1979a, 1979b).[23]

As it turns out, this axiom, which requires the simplification of a utility index which is "linear in the probabilities", simultaneously produces the unintended restriction

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[22] To a more limited extent this is also true of the generalized model used in chapter three.

[23] This issue of the timing of the resolution of uncertainty is important but tangential to the particular focus of this study, and therefore will not be explored further - interested readers are directed to the cited papers by Kreps and Porteus.

on time and risk preferences as a theoretical byproduct.[24] Unfortunately, the traditional axioms of the EU hypothesis are not known to reveal this interconnection between time and (conditional) risk preferences (Weil, 1990).

Using a simple two period framework, Selden (1978) and Rossman and Selden (1978) point out that the use of expected utility precludes consideration of a large and important class of time and risk preference specifications.[25] In particular, they show that not every perfectly standard ordinal intertemporal utility index $U(v_1, v_2)$, which captures the time preferences of an individual over known levels of first and second period indirect utility, and conditional utility index $V(\tilde{v}_2)$, which captures the risk attitude of the individual over an unknown level of second period consumption conditional on a given level of period one consumption, is compatible, either mathematically or economically, with the existence of a two period expected utility representation of preferences $W(v_1, \tilde{v}_2)$.

As an example of this problem, Rossman and Selden (pp. 70-71) show that behavioral anomalies such as risk preference reversals are likely in a two period EU model when first period consumption is changed and where (i) time preferences U are strictly quasi-concave and ordinal (such as in the CES class of preferences) and (ii) conditional second period risk preferences exhibit risk aversion. In fact, it is suggested by Rossman and Selden (although not proven) that any two period VNM representation of preferences which possesses constant EIS and CRRA parameters satisfying the following two conditions:

(i) $0 < \text{(EIS)(CRRA)} < 1$

(ii) $\text{EIS} < 1$

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[24] Selden (1978, 1979) and Rossman and Selden (1978) refer to the reduction axiom as the "coherence" axiom, and show that it essentially requires a special integration of an individual's set of conditional risk preferences and time preferences. As a result of forcing this integration on the preference ordering, incompatibilities emerge between otherwise reasonable utility functions. By relaxing the coherence axiom it becomes possible to expand the set of compatible preference specifications, as chapter three exploits.

[25] Although the authors cited define their particular indices over consumption bundles, this does not affect the generality of their claim.

will produce the unfortunate result that conditional risk preferences will be unstable for small changes in the level of first period consumption.[26] As a consequence, a researcher is justified in using a two period EU framework for comparing choices over lotteries only when a common level of first period activity exists (see also Dreze and Modigliani, 1972, p. 314).

When using a two period expected utility framework, three types of preferences exist. Paralleling the model in chapter three, and following the discussion in Selden (1978), the objects of choice are the ordered pairs (v,F) in the product set $S = v_1 \times F$, where $v_1 \in (0,\infty)$ is the set of certain indirect utility possibilities in period one, and F is the set of cumulative distribution functions on $v_2 \in (0,\infty)$, which represent elements of risky indirect utility possibilities in period two. Three types of preference-utility structures exist:

(i) time preferences are defined over certain (and only certain) indirect utility pairs $v_1 \times v_2$ described by the binary relation $\preceq^t$ on $v_1 \times v_2$ and represented by a continuous monotone ordinal index $U: v_1 \times v_2 \to \Re$ with $(v_1,v_2) \preceq^t (v_1',v_2')$ iff $U(v_1,v_2) \leq U(v_1',v_2')$.

(ii) a set of conditional risk preferences $\{ \preceq_v \mid v \in v_1 \}$ each defined over F and conditioned on a specific element of v_1 and representable according to the expected utility hypothesis. Define $V_v : v_2 \to \Re$ as a continuous strictly positive monotone second-period VNM utility index with $F_1 \preceq_v F_2$ iff

$$\int_{v_2} V_v(v_2)dF_1(v_2) \leq \int_{v_2} V_v(v_2)dF_2(v_2) \quad \forall F_1, F_2 \in F$$

where the V_v functional is unique up to a positive affine transform.

(iii) preferences over $S = v_1 \times F$, described by the binary relation $\preceq$ and represented by a continuous, strictly monotone two period cardinal utility index $W: v_1 \times v_2 \to \Re$ where

$$(v_1,F_1) \preceq (v_1',F_2) \text{ iff } \int_{v_2} W(v_1,v_2)dF_1(v_2) \leq \int_{v_2} W(v_1',v_2)dF_2(v_2)$$

where the W functional is unique up to a positive affine transform.

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[26] Rossman and Selden show how a small change in the level of first-period consumption can produce risk-preference reversals regarding risky period 2 consumption, in the sense that the model implies that an individual who is risk averse before the change can become risk loving after the change; in this sense second-period risk preferences are "conditional" on first-period activity.

Although risk attitude is assumed atemporal, the level of risk is not, since resources can be reallocated to smooth out income fluctuations and thus affect the amount of risk borne by the individual in period two -- as a result, observed behavior toward risk depends not only on risk attitude but also on first period activities. It is in this sense that second period risk preference is conditional on what happens in the first period.

In constructing the VNM index W from (i) a given ordinal utility index U and (ii) a conditional VNM utility function V_v, Pollak (1967), Selden (1978), and others have observed that the intertemporal index U and the two period VNM index W are closely related since they define the same indifference classes, and thus must be positive monotonic transformations of each other. (U, V_v)-pairs which, when combined to form W, do not produce this relationship between U and W are theoretically incompatible with the expected utility hypothesis.

The consequence is that there must exist a strong interconnection between time and risk preferences, in the following sense: consider computing the certainty equivalent (conditional) period two indirect utility values in the simple two state case

$$\hat{v}_2 = V_{v_1}^{-1} \int_{v_2} V_{v_1}(v_2) dF_1(v_2) = V_{v_1}^{-1} (\pi V_{v_1}(y_1) + (1-\pi) V_{v_1}(y_2))$$

$$\hat{v}_2' = V_{v_1'}^{-1} \int_{v_2'} V_{v_1'}(v_2') dF_2(v_2') = V_{v_1'}^{-1} (\pi V_{v_1'}(y_1') + (1-\pi) V_{v_1'}(y_2'))$$

where $(v_1, \hat{v}_2) \sim (v_1, F_1)$ and $(v_1', \hat{v}_2') \sim (v_1', F_2)$ given conditional risk preferences. As Selden (1978) points out, the expected utility hypothesis requires[27] that $(v_1, \hat{v}_2)$ and $(v_1', \hat{v}_2')$ lie on the same time preference indifference curve when

$$(v_1, y_1) \sim^t (v_1', y_1')$$

and

$$(v_1, y_2) \sim^t (v_1', y_2').[28]$$

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[27] via the reduction or "coherence" axiom.

[28] The indifference curve passing through $(v_1, \hat{v}_2)$ can lie above or below $(v_1', \hat{v}_2')$ in the model used in chapter three.

which obviously places significant restrictions on the form of V_v. In other words, for any given U, V_v must be chosen in such a way that U and W exhibit the same time preference properties - since U is ordinal it must be the case that U and W are positive monotonic transformations of each other in order to have a valid expected utility model. This restriction produces the incompatibilities regarding (U, V_v)-pairs due to the desire for the mathematical simplification of "linearity in the probabilities", or equivalently by adopting the reduction axiom.[29]

This axiomatic restriction on the joint form of preferences (U, V_v) is quite limiting for modeling purposes, especially when using the common time-additive, isoelastic specification of U appearing in the literature. It is easy to see that if one adopts the CES form for U that appears in chapter three

$$-\frac{1}{\delta_1}\left((v_1)^{-\delta_1} + \beta\ (v_2)^{-\delta_1}\right)$$

or its more common ordinal equivalent[30]

$$\left((v_1)^{-\delta_1} + \beta\ (v_2)^{-\delta_1}\right)^{1/\delta_1}$$

then the only isoelastic form of V_v which is compatible, in the sense described above, with U in the construction of the two period VNM index W is the power utility function[31]

$$-\frac{(v_2)^{-\delta_2}}{\delta_2}$$

where $\delta_1=\delta_2$. This means that the elasticity of intertemporal substitution is constrained to be equal to the reciprocal of the coefficient of relative risk aversion, since from either form of U the EIS parameter is

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[29] Again, the problem ultimately stems from the way expected utility attempts to represent two distinct preference concepts with a single, cardinal utility index.

[30] Note that the transformation $-\delta^{-1} (v_t)^{-\delta}$ is order-preserving for values of v_1 and v_2 here and in chapter three such that $v_1, v_2 \geq 0$. Positive values of v_1 and v_2 are guaranteed in chapter three for the reasonable preference specifications (i) $0< b < 1$ and (ii) $w_t, y_t > 0$ $\forall t = 1,2$, where b is the consumption-leisure preference parameter in a Cobb-Douglas utility function.

[31] Unsurprisingly, power utility functions normally characterize risk preferences in standard time-additive EU models (Weil, 1993, p. 369).

$$\frac{1}{\delta_1 + 1}$$

whereas from V_v the CRRA parameter is

$$\delta_2 + 1$$

If $\delta_1 = \delta_2$, as it must in order to form a two period cardinal VNM index W which is a monotone transform of U, it is clear that the two preference parameters become entangled.[32]

Since the derivation of closed-form optimization rules relies upon the use of isoelastic time preference and constant risk preference specifications, the cost of using an intertemporal VNM index for this purpose is to artificially restrict tastes in a way that essentially confuses the distinct preference concepts of risk aversion and aversion to intertemporal substitution. This meshing of attitudes obstructs a clearer theoretical understanding of important economic topics such as (i) the motives behind precautionary saving, (ii) the Ricardian equivalence debate, (iii) the permanent income hypothesis, (iv) the risk-free rate puzzle, (v) the covariance of equity and bond prices, as well as many empirical anomalies in economic data (see literature review below).

Though the arguments above do not prove that the expected utility framework is inadequate in all temporal settings, they provide a case for exploring more general utility functions.

Two Period GEU Preferences

As indicated earlier, a relaxation of the reduction axiom of expected utility theory permits the independent selection of the functional forms representing risk attitude and

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[32] The forms of W and U above will obviously be monotonic transforms of one another (they will in fact have identical forms). This will not be true with the preference specifications adopted in chapter three, where $V_v = -\exp(av_2)$. Thus, an expected utility representation of these preferences is not possible.

intertemporal preferences. Here we will examine the simple method used in chapter three which achieves this flexibility without sacrificing the goal of generating consistent preference orderings.

The two period ordinal certainty equivalent (OCE)[33] model of Selden (1978, 1979), which was the first to achieve the desired separation of preferences and has been used in a number of recent papers (e.g. see Barsky, 1989, Persson and Svensson, 1989, Werner, 1990, van der Ploeg, 1992, and Langlais, 1995), as well as the multiperiod model characterizing sequential decision making in Kreps and Porteus[34] (1978, 1979a, 1979b), involve a slight weakening of the VNM axioms. As a result of the empirical and theoretical difficulties associated with the reduction axiom of the expected utility hypothesis, an alternative preference structure which relaxes this restriction would seem desirable.[35]

OCE and Kreps-Porteus preferences both generalize the class of time-additive EU models by relaxing the VNM "reduction of compound lotteries" axiom, which effectively requires the use of an aggregator utility index that is linear in the probabilities. This preference constraint is violated, for example, by taking certainty equivalent level of future activities in the time aggregator function when the individual is not risk-neutral.

The OCE model as constructed in chapter 3 relies on an aggregator function U over period t indirect utility v_t and the certainty equivalent of period t+1 indirect utility

$$u_t = U\left[v_t, V^{-1}EV(\tilde{v}_{t+1})\right]$$

where u_t denotes current utility, E denotes an expectation operator and V denotes the index capturing conditional risk attitude in period two, to replace the common two-period

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[33] OCE generalizes the class of time-additive, VNM preference orderings in the two period case by relaxing the "coherence" axiom, which permits the use of independent functions to represent risk attitudes (V) and intertemporal tastes (U) while remaining time-additive.

[34] Kreps-Porteus preferences represent a time-consistent multiperiod generalization of OCE preferences and appear in a wide body of theoretical and empirical work (see, for example, Epstein and Zin, 1987, 1989).

[35] The reduction axiom is not the only troublesome axiom underlying the expected utility hypothesis. The independence axiom is also believed by many researchers to be a primary source behind the empirical difficulties of EU models, as the St. Petersburg paradox and Allais' paradox suggest. However, both OCE and Kreps-Porteus preferences retain this axiom as part of their structure.

cardinal EU function. Thus U is the function through which current and future indirect utility are aggregated, implying current utility is an aggregate, computed by using the function U[.,.], of current indirect utility and the certainty equivalent of future indirect utility.

The model is general in that it includes the EU specification as a special case (obtained by a reapplication of the reduction axiom) and also allows for the independent parameterization of time preferences (as determined by the curvature of U) and risk preferences (as determined by the curvature of V), thus breaking the link between aversion to risk and aversion to intertemporal substitution inherent in the EU framework.[36] This generalization of expected utility preferences is comparable to the adoption of the translog utility function in demand theory in order to relax the constraints imposed on the pattern of substitution across commodities by the more restrictive Cobb-Douglas and CES forms - it provides a more flexible and general analytical framework in which to perform theoretical and empirical research. Similarly, it would be useful to have more flexible functional forms that generate tractable solutions to economic problems.

However, while two period EU is a special case of the OCE representation of preferences, and although every EU representation can be transformed into an OCE representation, it is not true that every OCE representation can be transformed into an EU representation, since not every OCE representation is "linear in the probabilities". In general, the certainty equivalent representation of preferences over period 2 indirect utility will not be linear in the expectations operator unless V assumes a very restrictive functional form.

In chapter three a simple real stochastic two period partial equilibrium OCE model is developed and used to show how optimizing individuals respond to changes in the riskiness of several different income sources,[37] and how factor supplies are used to insure

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[36] This also stands in contrast with commonly used time-nonseparable models as well.

[37] Chapter three uses the simplest possible model of the consumption-saving problem: a two period model with additively time-separable utility with income risk in period two only.

against adverse outcomes (see Block and Heineke, 1972, 1973, Sellin, 1989, and Werner, 1990). The comparative static results demonstrate the important role that the EIS parameter performs in determining the slope of the income time profile and thus the level of precautionary saving that occurs in the face of income risk. The essence of the precautionary saving motive is the way income risk has of tilting, relative to the certainty case or the risk neutral case, the income time profile toward the future.

The model in chapter three is neoclassical, in that its foundation rests on the perfect-market paradigm of rational behavior and frictionless, competitive, and informationally efficient markets. The preferences used can also be considered Kreps-Porteus, which is equivalent to OCE preferences in a two period environment, and are similar to preferences appearing in a model by Persson and Svensson (1989).

Two alternative insurance mechanisms are available to the representative individual in chapter three to smooth income over time: (i) labor supply variations after the shock to income has occurred and (ii) precautionary saving variations before the shock to income has occurred. The presence of either univariate or multivariate future income risk will always give motivation to use second period labor supply as a form of ex post insurance, and may also result in the use of first period precautionary saving as a form of ex ante insurance when the EIS is low.

The model differs from most previous literature by (i) incorporating leisure as well as consumption in the direct utility function,[38] (ii) forming the certainty equivalent of future activities over an indirect utility function (instead of over consumption or utility of consumption), (iii) considering multiple sources of income risk, and (iv) deriving closed-form solutions. It also differs by using a negative exponential utility index defined over risky levels of future income instead of the more common isoelastic one.[39]

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[38] Dardanoni (1988) considers a completely general two-good model of behavior under uncertainty. The use of indirect utility as arguments in chapter three can be considered a special case of the work by Dardanoni.

[39] Since the two period model in chapter three assumes time-additive preferences, the risk aversion function is defined only over second period indirect utility.

The use of indirect utility functions as temporal arguments is useful for modeling effects without overburdening the model with the detail of the primitive preferences.

In general it is not obvious what risk concept to use when preferences are specified over several goods (see Hanoch, 1977). The model used here bypasses this problem by assuming a semi-recursive formulation which defines time preferences by an intertemporal utility index over a known level of first period indirect utility and a certainty equivalent level of unknown second period indirect utility. This formulation is closer to a recursive structure than the original OCE model of Selden (1978) defined over period consumption levels.

However, time consistency problems still occur when this model is used in a multiperiod framework[40] since the second period indirect utility function does not represent the second period evaluation of the entire future path of indirect utilities (which is automatically true in the simple two period case). In this sense the model does not possess the truly recursive structure

$$u_t = U(c_t, \tilde{u}_{t+1})$$

which appears in, for example, Kreps and Porteus (1978, 1979a, 1979b), Epstein and Zin (1989, 1991), and Farmer (1990).

To clarify this point, consider the seemingly natural algorithm for computing utility

$$U(\tilde{v}) = \sum_0^\infty \beta^t u(\hat{v}_t)$$

$$\hat{v}_t \equiv V^{-1} E_0 V(\tilde{v}_t)$$

where $\tilde{v}_t$ represents random indirect utility in period t, $\hat{v}_t$ represents the certainty equivalent level of period t indirect utility, and V is the utility function defining risk attitude. Thus, the random indirect utility is replaced by its certainty equivalent in the intertemporal aggregator function U, and the intertemporal utility of the sequence of certainty equivalents is computed in a simple additive way. Risk attitude can be changed by an appropriate reformulation of the function V while keeping β and u fixed.

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[40] As realized by Hall (1985), Zin (1987), and Attanasio and Weber (1989).

However, the function violates weak recursivity in the sense that preferences for current and future levels of indirect utility will depend on past unrealized alternatives, as discovered by Hall (1985), Zin (1987), and Attanasio and Weber (1989).[41] This is seen by noting that period t preferences would continue to be based upon certainty equivalents computed back in period 0 (i.e. the expectations operator E_0 is used in the algorithm).

In addition, an assumption that the certainty equivalents are updated each period implies that preferences exhibit intertemporal inconsistency, and suggests the behavior of a naive individual who ignores inconsistencies in her selected indirect utility program over time and continually revises her plans.

The model in chapter three assumes hybrid preferences, in that the ordinal time aggregator is generalized CES in form while period two risk preferences are assumed negative exponential and thus exhibit constant absolute risk aversion.[42] The single parameter determining the curvature of the time preference function is constant as a result, which facilitates the derivation of closed-form optimal decision rules. If the aggregator were not assumed isoelastic the EIS parameter would become a function of the specific level of indirect utility in each period, whereas if risk preferences in period two did not assume linear risk tolerance the coefficient of absolute (or relative) risk aversion would become a function of intertemporal wealth - neither specification would yield closed-form optimal decision rules.

The ordinal time aggregator U in chapter three also assumes time-separable preferences, implying that past work and consumption do not directly influence current and future tastes, but indirectly do so to the extent that they show up as current state variables, such as wealth, in the budget constraint, rather than the more realistic assumption of time-

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[41] "Weak" recursiveness is obviously a weaker property than the state-separability exhibited by intertemporal expected utility functions - together with the assumption of constant tastes it represents a fundamental assumption in both expected utility theory and many generalizations of it (Epstein and Zin, 1989).

[42] Extensive use of Taylor approximations in chapter three reveals that constant absolute risk aversion corresponds locally to the standard mean-variance approach in finance for the special case of normally-distributed random variables.

nonseparable preferences for a number of reasons. Most importantly for the purposes here, a time-nonseparable framework will not allow for the clean separation of time and risk preferences (Epstein and Melino, 1995).

Barro and King (1984, p. 835) also suggest that using time-nonseparable preferences, such as habit-persistence models, do not guarantee strong testable restrictions that are usually generated by time-separable preferences.[43] They note that much of the empirical attraction of the permanent income hypothesis is derived from the treatment of past consumption as bygones which are unimportant for current decisions. This will be important when the empirical content of a recursive version of the two period OCE model is tested in chapter five for asset pricing purposes. The structure of the model makes it easy to test it with historical consumption and market return data.

Furthermore, to yield tractable and analytically convenient closed-form solutions from the model, a time-separable form of preferences is necessary.[44] An even stronger preference assumption, time-additivity, is imposed in the model so that the study of the demand for consumption and the demand for leisure can be conducted independently in each period.

Finally, it seems that departures from separability matter more over days and weeks rather than months, quarters, or years, so that if the two time periods are interpreted as covering long intervals, the convenient abstraction of separability becomes more palatable.[45]

The form of the risk preference function used in chapter three belongs to the family of utility functions with hyperbolic absolute risk aversion (HARA) preferences, which includes the common quadratic, logarithmic, and isoelastic (or power) utilities as special cases. This family of preference representations is particularly useful for obtaining

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[43] The importance of this issue is raised in chapter 5 in the context of empirical asset pricing research.

[44] A distinguishing feature of time-nonseparable stochastic modeling is the cumbersome way in which analytical results are obtained, if at all (Attanasio and Weber, 1993).

[45] However, the existence of time-nonseparabilities in the utility index make current consumption (especially of nondurable goods and services) a poor proxy for wealth in asset pricing applications. Weil (1990) comments on this problem in multiperiod applications.

reduced-form solutions for optimal decision rules, and are the only time-additive and independent preference orderings that lead to optimal saving functions which are linear in wealth (Ingersoll, 1987).[46]

A nice attribute of the particular preference specification assumed in chapter three, isoelastic intertemporally but exponential in its risk dimension, is that it guarantees that negative consumption levels will not occur, since the marginal utility at zero consumption levels will be infinite (Weil, 1993, p. 368). Thus we can rule out empirically unappealing corner solutions given the model setup.

It would clearly be desirable to analyze intertemporal factor allocation under uncertainty in a model with multiple time periods. However, one cannot simply set up a multiperiod model similar to the one appearing in chapter three and use the same calculus to obtain results - such a model would unrealistically imply that all future saving is determined in the current period. Actual saving is sequential, so that the dynamic programming would be a suitable tool for analysis, as in Kreps and Porteus (1978, 1979a, 1979b). However, solving these dynamic models are mathematically formidable when uncertainty is incorporated, and unless one is willing to make extremely restrictive preference assumptions (e.g. log utilities), the goal of deriving analytically tractable, closed-form decision rules, as is done in chapter three, would be difficult.

Literature Review

The theory of consumption/saving decisions under uncertainty has recently seen a revival in the theoretical literature, following two main directions: (i) showing that

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[46] A CARA risk preference structure implies that no endogenous change in the level of risk currently undertaken will occur due to a change in wealth; in other words, risk has no effect on the marginal propensity to consume. This is a strong preference restriction, given that empirical observations make "decreasing" absolute risk aversion all but an obligatory modeling assumption. However, CARA has the advantage of producing clean, tractable analytic solutions to many stochastic, intertemporal problems.

"prudence" is a distinct preference concept from risk aversion (see Kimball, 1990, and Eeckhoudt et al., 1995) to refine precautionary saving theory under expected utility[47], and (ii) using the precautionary saving motive to help resolve some empirical consumption puzzles[48] (see van der Ploeg, 1993, and Weil, 1993) or in explaining the failure of Ricardian equivalence (see Barsky, 1989, and van der Ploeg, 1992) under nonexpected utility.

The move away from expected utility modeling of precautionary saving behavior in the theoretical literature began in the late 1970's with the publications of Selden (1978) and Kreps and Porteus (1978, 1979a, 1979b).

Using the basic two period OCE framework derived in Selden (1978), Selden (1979) discusses the effect of capital income risk on precautionary saving behavior, and concludes that the level of risk aversion is irrelevant to determining whether saving rises or falls with the introduction of capital income risk, and that the EIS is the critical parameter determining saving behavior.[49]

A similar result is obtained in Werner (1990) and Langlais (1995). Using an OCE representation of preferences in a general equilibrium model of international capital markets, Werner also examines saving and labor supply behavior when the return to saving is risky due to capital controls, and shows how these factor supplies are used to insure against future income risk. Sellin (1989) examines this issue in a stochastic continuous-time framework.

Other recent literature incorporating the OCE preference framework includes the empirical studies of Hall (1988), Barsky (1989), and Obstfeld (1994a, 1994b), and the theoretical work of Persson and Svensson (1989), van der Ploeg (1992), and Weil (1993).

Hall (1988) discusses the importance of distinguishing between risk aversion and intertemporal preferences in macroeconomic modeling and uses an OCE framework to

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[47] There is some complementarity between the expected utility notion of prudence and a low elasticity of intertemporal substitution, although the mathematical concepts are distinct - see van der Ploeg (1993).
[48] e.g. issues involving the permanent income hypothesis and asset markets (Weil, 1990).
[49] This result resurfaces in chapter three in a modified form.

empirically estimate the EIS of a representative U.S. consumer given U.S. consumption and real interest rate data. His results suggest that extreme aversion to intertemporal substitution characterizes such an individual.[50,51]

Persson and Svensson (1989) use OCE in a two period, two country general equilibrium analysis of international portfolio choice to determine what international capital flows arise from intertemporal preferences and what ones arise from risk aversion. They conclude that if the home country is more averse to risk and intertemporal substitution relative to the foreign country, it will import indexed (risk-free) bonds from the foreign country to increase period two consumption and decrease period one consumption. The import of indexed bonds will dominate the export of risky bonds, the home country will have a positive net foreign investment, and the home country's capital (current) account will be in surplus (deficit) in a free exchange regime.

Another use of Selden's OCE framework is found in Barsky (1989), who attempts to answer the question of why stock and bond prices in the U.S. do not move together over time. Similar to the findings in chapter three, Barsky comments that net stock prices may rise or fall given an increase in capital return risk depending on the value of the EIS parameter, in that a higher (lower) EIS value results in lower (higher) equity demand and equity prices when capital return risk rises, for any given level of risk aversion.[52] This

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[50] Hall's findings support an EIS value of about 0.1, given U.S. data. In addition to Hall, other researchers have pointed to the fact that the standard EU model would infer from this that risk aversion, as well as aversion to intertemporal substitution, must be extremely high in the U.S. This conclusion finds scant support in the empirical literature.

[51] Hall bases his results on the fact that aggregate consumption growth and aggregate saving rates in the U.S. do not appear to be terribly responsive to changes in real interest rates. However, given his reliance on aggregate data as well as an implicit assumption of complete capital and credit markets, it is sensible to conclude that his results do not provide conclusive evidence on the magnitude of the EIS parameter in the U.S. (Weil, 1993). The current literature would support virtually any EIS parameter value less than one, with values between 0.3 and 0.8 appearing to be most reasonable (Attanasio and Weber, 1993), although Beaudry and van Wincoop (1994) claim to find evidence of a value close to one using U.S. panel data.

[52] In particular, Barsky notes in the two asset case that equity return will fall (rise) as capital risk rises when the EIS parameter is less (greater) than roughly 0.2. He goes on to say (p. 1141) that although the CRRA parameter affects the magnitude, in either direction, of the effect of capital risk on equity returns, it has "...no bearing on the sign." Implicitly, he is assuming risk averse behavior to justify this statement, as chapter three points out.

stands in contrast to riskless bond demand and prices, which unambiguously rise when capital return risk increases, so that a fall in the riskless rate is a more robust prediction to offer when financial markets become volatile than a rise in equity returns.

Thus, to get stock and bond prices to covary negatively, a relatively low aversion to intertemporal substitution must first be assumed in the model.[53,54]

Using an OCE framework to reinterpret the Ricardian propensity to consume out of a current tax cut, van der Ploeg (1992) notes that in the presence of proportional income taxation and future income risk, and in the absence of trade in claims on human wealth, even an individual with Ricardian rationality will increase consumption when current wage taxes are cut if her risk aversion is high enough and her aversion to intertemporal substitution is low. This occurs since certain current income is created by the tax cut while the associated increase in future income tax rates will reduce the variability of future after-tax labor return. A risk averse Ricardian who does not mind intertemporal consumption substitution treats the temporary boost in real wages as a "bird in the hand" by increasing current consumption and reducing saving.[55]

This indicates that, in addition to factor supplies, a future tax increase which is proportional to income can provide insurance when future income is uncertain which otherwise would be provided in a complete contingent claims market where labor income risk could be diversified away.

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[53] In a one asset model Barsky finds that an unrealistically high EIS value (> 1) is necessary to explain the puzzle. However, in a two asset model with a single financial asset and a single real asset, he derives the result that increasing capital return risk coupled with decreasing productivity in the corporate sector of the economy will simultaneously cause stock prices to fall and risk-free bond prices to rise for reasonable EIS values (≈ 0.2). Barsky notes that the explanation in the popular financial press of opposite stock and bond price movements suggests a "flight to quality" argument by nervous, increasingly risk averse investors - such an explanation finds little support from either EU or OCE theory, based as it is on the second derivative properties of the period utility function. Rather, a rise in precautionary saving in less risky assets involves the third derivative of the utility function under EU and intertemporal preferences under OCE.

[54] However, this does not imply a resolution of the so-called "equity premium puzzle", which depends on risk aversion alone. This is not surprising since the equity premium puzzle is well-defined even in a model without intertemporal choice (Barsky, p. 1141). Additional discussion of the equity premium puzzle is found in chapter five.

[55] This finding can be compared fruitfully to the results obtained in chapter three with stochastic real wages.

Van der Ploeg also observes that when the EIS and CRRA parameters are constrained to be reciprocals of one another a wage tax cut will have no effect on current consumption behavior (i.e. standard Ricardian behavior results). However, when the constraint on preferences is broken, then two cases emerge when a tax cut occurs: (i) Keynesian consumption behavior is adopted by a Ricardian individual when the CRRA is greater than 1/EIS, as suggested above and (ii) current consumption actually falls and saving rises for a Ricardian individual when the CRRA is less than 1/EIS.

Reinterpreting the permanent income hypothesis, Weil (1993) uses the OCE framework to examine labor income uncertainty and the implications for precautionary saving. While more prudent behavior implies a partial hedge against uncertainty, thus satisfying the risk averse side of an individual's personality, it also generates higher income in period two, which runs counter to the impulse to smooth consumption over time. Weil finds that a weaker desire to smooth income (i.e. a higher EIS parameter) results in more prudent behavior from a risk averse individual. He also notes that prudent behavior is reinforced by larger income risk, stronger aversion to risk, and higher interest rates.[56]

Obstfeld (1994a) uses a recursive preference representation in a dynamic stochastic model to separate the roles of risk attitude and intertemporal preferences to evaluate the welfare costs of consumption instability in the U.S. as well as the potential welfare benefits of pursuing consumption stabilization policy goals in government.

He indicates that while stronger risk aversion will increase the period cost of consumption instability, stronger aversion to intertemporal substitution will increase the cumulative cost of consumption instability that persists over time, and thus is an important preference parameter in dynamic welfare analysis.

However, in time-separable EU preferences, an increase in risk aversion will change the welfare cost of consumption instability in ways that are unrelated to risk

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[56] An interesting observation from this work is that individuals with extremely low levels of wealth will probably engage in precautionary saving behavior that is a function of wealth, whereas such behavior for high wealth individuals will be independent of wealth.

attitude. The confusion of preferences there creates a misleading picture of the importance of risk attitude, and causes the welfare cost of consumption instability to appear much lower than is the case when the two preference concepts are treated as distinct.

Along a similar path, Obstfeld (1994b) uses the same recursive class of isoelastic preferences in a dynamic stochastic model of international trade in assets to evaluate the potential welfare gains from the increased international risk-sharing that would result from the elimination of international capital controls.[57]

The classical expected utility literature on precautionary saving under uncertainty (e.g. Leland, 1968, Samuelson, 1969, Levhari and Srinivasan, 1969, Sandmo, 1970, Block and Heineke, 1972, 1973, Dreze and Modigliani, 1972) has been concerned with individual responses to mean-preserving increases in income (particularly capital income) risk, which are the relevant changes in the distribution of a random variable for a proper understanding of risk attitude according to Rothschild and Stiglitz (1970, 1971) and Diamond and Stiglitz (1974). These types of shifts in distribution are used in chapters three and four.

A standard EU result with time-additive preference is that more uncertainty about future income induces precautionary saving if $U'''( ) > 0$ (i.e. if the marginal utility of consumption is convex over the risky variable (see Leland, 1968, Sandmo, 1970, Kimball, 1990, and van der Ploeg, 1993), while risk preferences with zero third derivatives (e.g. risk neutrality, quadratic) do not give rise to precautionary saving behavior.[58]

Another standard result in the EU literature is that a mean-preserving change in capital income risk will induce greater precautionary saving only if the single parameter of the time-additive isoelastic utility index is less than unity (i.e. the intertemporal utility index is more concave than the logarithmic case (see Samuelson, 1969, Merton, 1969, and Levhari and Srinivasan, 1969).

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[57] The recursive preference representation that Obstfeld uses in both of his cited papers is drawn from Epstein and Zin (1989) - an evaluation of the Epstein-Zin class of preferences for asset pricing applications is the topic of chapter five.

[58] Any utility function exhibiting decreasing absolute risk aversion (e.g. a negative exponential risk preference function) implies that a convex marginal utility function exists and that individuals will engage in precautionary saving.

Thus, time-additive EU models imply that precautionary saving in response to a change in income risk is associated with the curvature of the marginal utility function.[59] However, by introducing risk attitude in the felicity function this work does not distinguish between risk aversion and aversion to intertemporal substitution. In fact, the term "elasticity of intertemporal substitution", or its equivalent, rarely appears in the EU literature at all.[60]

Leland (1968) uses a simple two period model to argue that two possible assumptions exist which guarantee that an individual will engage in precautionary saving as exogenous income uncertainty increases, one of which is that a time-additive utility index over consumption must exist which exhibits temporal decreasing absolute risk aversion (DARA) over both periods, which essentially means that aversion to risky period two consumption decreases as the level of period two consumption increases; i.e. the third derivative of the aggregator index U with respect second period consumption (c_2) is positive

$$\frac{\partial^3 U(c_1, c_2)}{\partial c_2^3} > 0$$

This condition implies that the marginal utility of period two consumption is convex over the level of period two consumption.

Sandmo (1970) derives a similar result for a wage income shock, but shows that precautionary saving behavior is ambiguous when capital income is stochastic, even with the time-additive DARA preference assumption, since an increase in saving increases both the mean and the variance of future income.[61]

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[59] However, as is explicitly demonstrated in chapter four, these conclusions regarding the specific curvature of the utility index and the effect on precautionary saving are sensitive to the presence of other income sources beside capital income.

[60] As noted earlier, Dreze and Modigliani (1972) mention this concept indirectly in their work on capital return risk and saving behavior.

[61] Note here that Sandmo treats labor income as exogenous, unlike the model in chapter three.

Samuelson (1969) and Levhari and Srinivasan (1969) extend this additional result of Sandmo and use multiperiod, isoelastic[62] EU models to show that a change in capital return risk will either raise or lower saving depending upon whether the marginal utility of consumption is greater or less than unity in its elasticity. Thus, the Bernoulli case, in which relative risk aversion is constant and unitary, emerges as a watershed between instances where thrift is enhanced by riskiness rather than reduced.

In a companion paper to Samuelson's, Merton (1969) demonstrates that this result is robust in a continuous-time framework.[63]

Additional work using expected utility theory that is relevant to us here includes the papers by Block and Heineke (1972, 1973), who use a VNM framework to study labor supply responses to changes in the riskiness of (i) a random real wage and/or (ii) a random future endowment.[64] In their model individuals use first period labor supply to hedge against uncertainty, and they note that additional income sources should reduce saving response to income uncertainty.

Two complementary papers that examine the effect of exogenous wage income uncertainty on precautionary saving behavior are by Sibley (1975) and Miller (1976), both of whom extend the two-period model of Sandmo (1970) into a multiperiod framework. They derive results generally supportive of Sandmo's observation that saving behavior under wage income risk will depend crucially on the curvature of the marginal utility function over the random variable, in that saving will be augmented (reduced) as the marginal utility function becomes more convex (concave).[65]

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[62] Assuming constant relative risk aversion, as Samuelson does, is equivalent to assuming that the marginal utility of the risky variable (e.g. second period consumption) is isoelastic.

[63] Dreze and Modigliani (1972) show in a simple two period model that with a perfect capital market assumption, where all income uncertainty is assumed endogenous, the impact of uncertainty on saving depends on whether the marginal rate of substitution between first and second period consumption is convex or concave over second period consumption, not on the elasticity of the marginal utility function. They also generalize the results of Leland (1968) and Sandmo (1970), who only consider infinitesimal risk changes, to cases of large risk change and its effect on saving.

[64] But without a second period labor-leisure choice and no stochastic capital returns.

[65] Sibley also goes on to consider the effects of assuming various stochastic processes for dynamic wage movements on the classic permanent income hypothesis (PIH) theories of Friedman. He shows that a

More recent contributions to the literature that use expected utility frameworks include the multiperiod work of Zeldes (1989), Kimball and Mankiw (1989), and Caballero (1990), as well as the two period models of Kimball (1990) and Eeckhoudt et al. (1995).

Zeldes (1989) derives an n-period aggregate consumption function using U.S. panel data estimates of wage income uncertainty to investigate some well-known consumption puzzles. A conclusion he draws from this work is that low current saving will induce a relatively large precautionary saving response to an increase in the riskiness of labor returns, suggesting that precautionary saving may constitute a significant share of total aggregate saving in the U.S. economy.

Kimball and Mankiw (1989) use an infinite-horizon framework to investigate the insurance effects of future labor income tax increases on saving behavior similar to the work of van der Ploeg (1992) discussed earlier, as well as the response of saving to the timing of these taxes. They show how a proportional wage tax regime results in the failure of Ricardian equivalence due to the insurance effects of potential future taxation, which reduces the variability of future labor return and thus for a risk averse individual encourages higher current consumption, lower current precautionary saving, and predictable consumption/saving behavior over time.

Caballero (1990) uses a similar model to Kimball and Mankiw to show how an explicit consideration of both the precautionary saving motive and the stochastic processes driving labor returns can resolve several puzzles in modern consumption theory.

In an interesting extension of the EU tradition that mirror the classic work of Pratt (1964) on risk attitude, Kimball (1990) and Eeckhoudt et al. (1995) discuss the behavioral notions of "prudence", exhibited by a utility index with convex marginal utility over risky future consumption and saving

$$\frac{\partial^3 U(c_1, c_2)}{\partial c_2^2 \partial s} > 0$$

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homoscedastic wage process assumption is consistent with the PIH, but that heteroscedastic assumptions are not consistent with it.

and suggesting the degree of sensitivity of the level of optimal saving to risk, and

"temperance", exhibited by a utility index with decreasing absolute prudence in

consumption and saving

$$\frac{\partial\ ^4 U(c_1, c_2)}{\partial\ c_2^2 \partial\ s^2} > 0$$

and indicating the degree to which the precautionary saving motive falls as both risk and

saving levels rise.[66]

Kimball notes that (i) risk-aversion ($U''(\ ) < 0$), which indicates a dislike of mean

preserving increases in risk, (ii) prudence, which indicates a preference for mean-

preserving increases in risk which skew more risk into wealthier states and less risk

elsewhere, and (iii) temperance, which indicates a dislike of any skewing of the density

function following a mean-preserving spread, can all be interpreted as systematic attitudes

toward statistical transformations of the density functions of random variables.

He goes on to show that prudence is a necessary and sufficient condition for

individuals to prepare and forearm themselves by saving more in the presence of

uncertainty, closely paralleling the earlier work of Leland (1968) and Sandmo (1970). It is

also demonstrated that temperance will result, for example, in an individual reducing

demand for a risky asset when other independent income sources become riskier.

In this context, the marginal utility of consumption plays the same role for

precautionary saving that the utility function itself plays for risk aversion, with the shifts

and swivels of the consumption function for an individual in response to income risk being

a good measure of the level of prudence.[67] The consumption function shifts right (left) for

a prudent (nonprudent) individual when income risk increases. The precautionary

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[66] Prudence indicates that an individual becomes less worried about future income risk as precautionary saving efforts increase, and temperance indicates that an individual becomes more concerned about exposing additional current resources to future risk as precautionary saving levels rise. Effectively, prudence and temperance define the curvature of the marginal utility function for future risky consumption (or indirect utility).

[67] Kimball (p. 68) notes that "...the precautionary saving motive is risk aversion of the negative of marginal utility [of future risky consumption]." It is in this sense that he mirrors the work of Pratt (1964), who characterizes risk attitudes over risky outcomes by the second derivative of the utility function.

premium is decreasing (increasing) with the level of saving when decreasing (increasing) absolute prudence is exhibited, with the result that the consumption function swivels counterclockwise (clockwise) around a fulcrum that moves to the right (left) as income risk increases. For the singular case of constant absolute prudence ($U''' = 0$), the consumption function first shifts left, then right, in response to increases in income risk.

Recursive GEU and Asset Pricing

The problem of confusing time and risk preferences introduced by using a time-additive VNM index to model asset prices was mentioned by Lucas (1978, p. 1441), who noted that in such a framework there is "...no way to disentangle these conceptually distinct aspects of preferences."

As Mehra and Prescott (1985) have demonstrated, representative agent optimizing models which use a conventional time-additive and homogeneous VNM intertemporal utility index do not perform well in explaining the stylized facts of consumption and asset market data in the U.S. They stress how difficult it is to quantitatively account even for the average levels of asset returns, let alone their changes over time, with the asset pricing models currently in use by researchers, such as the C-CAPM.

A number of possible explanations have arisen to explain this failure: (i) the specification of preferences is too rigid (as discussed earlier), (ii) the absence of liquidity constraints, (iii) the absence of transaction costs, (iv) the assumption of complete markets, (v) the assumption of homogeneous agents by the use of a representative agent, (vi) the misspecification of the dynamics of market fundamentals, and (vii) the assumption that the distribution of expenditures across population cross-sections is constant over time (Attanasio and Weber, 1993). Assumptions (v) and (vii) deal with the aggregation biases inherent in many neoclassical models with representative agents.

However, as discussed briefly above, the simple two period model used in chapter three cannot be used in a multiperiod environment because of the lack of a truly recursive structure - such preferences may lead to intertemporal inconsistencies. This limits its application in a wide variety of economic areas, such as asset pricing.

In the context of the static CAPM and intertemporal consumption-based capital asset pricing model (C-CAPM), a recursive[68] extension of OCE preferences has been developed and proven to exist on a broad domain by Epstein and Zin (1989, 1991)[69] of the form

$$U_t = \left(c_t^\rho + \beta \left(E_t \, \tilde{U}_{t+1}^\alpha \right)^{\rho/\alpha} \right)^{1/\rho} \quad \forall \; t \geq 0$$

which can be used for asset pricing purposes, and where the parameters α and ρ are used in the representation to distinguish time and risk preferences. Unlike the simple two-period utility structure, which will generate Euler equations that apply only to naive individuals who continually ignore that their current plans will in general not be carried out in the future, this recursive utility structure produces intertemporally consistent (in the sense of Johnson and Donaldson, 1985) and stationary (in the sense of Koopmans, 1960) preference orderings.[70]

In addition, the Euler equations that result from this recursive utility structure involve the use of observable market data, as well as providing a clean separation of the

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[68] "Recursive" preferences are independent of all past realized and unrealized indirect utility levels, and thus represent a more restrictive constraint on behavior than the assumption of "weakly" recursive preferences defined earlier in the chapter. The assumption also implies the stationarity of preferences as well, so that stationary dynamic programming techniques are available for specified optimization problems such that those state variables which reflect past indirect utility realizations are unnecessary for modeling current behavior (Epstein and Zin, 1989). The assumption of either recursivity or weak recursivity facilitates the derivation of tractable solutions, but the former is clearly more useful for such a purpose. Note that intertemporal expected utility models only assume weakly recursive (but not recursive) preferences while OCE models generally assume neither structure.

[69] As noted earlier, this recursive model is constructed from a utility index of the form $U_t = U(c_t, \tilde{u}_{t+1})$

[70] These preference concepts were defined earlier in the chapter.

time preference (ρ) and risk preference (α) parameters of the individual that is absent in virtually all intertemporal EU and time-nonseparable models.[71]

Using such a model, Epstein (1988) shows how the EIS and CRRA preference parameters work together to determine current consumption and asset prices in the face of perceived capital return risk.

The EIS parameter is shown by Epstein to be the critical parameter in determining asset price movements.[72] Paralleling Barsky (1989), Epstein finds that if the EIS is greater (less) than unity in his model, then the substitution (income) effect of a rise in perceived saving return dominates behavior, saving levels and asset demand rise (fall), which in turn causes asset prices to rise (fall) and asset returns to fall (rise) until an equilibrium is restored.

On the other hand, the CRRA parameter only affects the certainty equivalent evaluation of future capital income, so that an increase in risk aversion reduces the certainty equivalent return to saving - if the EIS parameter is less than unity, the dominant income effect implies reduced current consumption, higher saving and asset demand and prices, and lower capital returns, while if the EIS parameter is greater than unity, the dominant substitution effect implies a current consumption binge (to avoid exposing certain income to future risk), lower saving and asset demand and prices, and higher capital returns.

As is clear from Epstein's discussion, a disentanglement of time and risk preferences provides a clearer and more intuitive explanation of asset market behavior than what is possible in a standard EU framework.

However, Epstein and Zin's recursive GEU approach to asset price modeling has a number of critics. Weil (1990) claims that simply modifying preferences will probably not replicate the intertemporal pattern of consumption and asset returns more accurately than

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[71] Again, Lucas (1978) must equivocate as to which aspect of utility, substitutability or risk aversion, determines the income sensitivity of asset prices in his EU model.

[72] Hall (1988, p. 339) states that one of the most important determinants of the response of saving to the real interest rate is the EIS parameter - to the real interest rate one could also add income uncertainty.

time-additive EU models.[73] He also criticizes the necessary assumption of time-separable preferences in the Epstein and Zin model, implying that nonseparabilities across time, states, and commodities are important (although he then goes on to say that time-separable preferences are probably not the most serious misspecification of asset-pricing and other macroeconomic models).

These criticisms are tentatively addressed in chapter five, where a first-level diagnostic tool is used to assess the empirical performance of Epstein and Zin's recursive generalization of the OCE preference representation using historical asset and market return data in the U.S.

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[73] Kocherlakota (1990, 1995) and Dardanoni (1993) suggest that separating time and risk preferences in a recursive asset pricing model will generate the same Euler equations as a standard C-CAPM model which uses an EU preference representation. However, this result is extremely sensitive to the specific modeling assumption found in these papers and is not valid for the Epstein and Zin model.

CHAPTER 3: GEU AND PRECAUTIONARY SAVING

Basic Model Features

The basic GEU model used in the analysis of this chapter, which assumes that preferences are exponential in their risk dimension and isoelastic intertemporally, is a stripped-down version of a more elaborate one used by Werner (1990). Some modifications from her model include specifying a second-period endowment income level, the use of a simpler and more manageable aggregator function, and a removal of the general equilibrium features which unnecessarily clutter her analysis, as mathematical appendix A demonstrates.

The model imagines a representative individual in an economy in which two sources of utility exist: a single consumption/investment good and leisure. This feature contrasts with most GEU literature which only considers a single argument in the utility function. The intertemporal environment is modeled as two distinct periods: the "current" period 1 and the "future" period 2. The second period is characterized by income uncertainty. No effort is made to identify the primitive, underlying source of the uncertainty in the modeling so that this cannot be considered a general equilibrium analysis of behavior under uncertainty. Income in period 1 consists of two types: endowment (e_1) and real wage/labor $(n_1 w_1)$ income, where the subscripts denote the time period, "n" represents labor supply, and "w" represents the real wage. Income in period 2 consists of three types: endowment (e_2), real wage/labor $(n_2 w_2)$ and real interest/capital $(s(1+r))$ income, where "s" represents first-period savings of the consumption/investment good by the individual, and "r" is the second-period real interest rate. It is assumed that the endowment income in both periods is received exogenously, so that the individual cannot modify her behavior in any way to enhance this income source. Real wage/labor income in each period is assumed to depend on two factors: (1) the endogenously-determined labor

supply of the individual during the period, and (2) the exogenously given real wage rate during the period. Second-period interest/capital income also depends on two factors: (1) the endogenously-determined level of saving in period 1, and (2) the exogenously given real interest rate in period 2.

Thus the individual has two decisions to make in period 1: the amount of labor to supply to the labor market and the level of savings (which de facto determines the level of first-period consumption in the model). The individual has only one decision to make in period 2: the amount of labor to supply to the labor market. At the end of period 2 the individual is assumed to die so that all available units of the single good available to her are consumed; i.e. no bequest motive is considered in the basic model.

When uncertainty exists in period 1 regarding period 2 income, it is assumed that only the first two moments of the distribution of any period 2 random variable(s) are subjectively formulated by the individual to guide productive efforts in period 1.[1] Furthermore, it is assumed that the individual understands that her behavior in period 1 cannot change these subjectively-perceived moments in any way. Both factor prices (as well as endowment income) in period 2 are assumed beyond the control of the individual, which is a plausible assumption to make if factor markets are perfectly competitive. This feature further distinguishes the model from Werner's, in which only endogenous risk is considered.

As we will see, the model predicts that consumption, saving, and labor supply should all covary over states and that patterns should emerge which reveal underlying preferences in the economy.

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[1] If the random income variables were assumed to have either a normal or lognormal distribution, this would be a necessary assumption to make since only two moments exist for these distributions. However, since Taylor approximations are used to deal with income uncertainty in the model, specific distributional assumptions are unnecessary.

Individual Preferences and the Budget Constraint

The model is a two-period version of the Kreps-Porteus framework, similar to the models of Selden (1978) and Farmer (1990) except that indirect utility levels are used as arguments in the temporal aggregator function.

The direct utility function of the representative individual is assumed to be Cobb-Douglas of the form

$$u_t(c_t, (1-n_t)) = c_t^b (1-n_t)^{1-b} \quad \text{for } t=1,2$$

where "c" represents the consumption level of the single good. This is a convenient function to use since it yields a particularly simple form of indirect utility function to use for analysis

$$v_t(y_t, w_t) = A w_t^{b-1} y_t \quad \text{for } t=1,2 \qquad \text{(see appendix A)}$$

where "y" represents income.[2] Of course a different direct utility function could have been specified (e.g. CES) but a more complicated form of indirect utility would result without special restrictions on preferences.

The income level in each period is assumed to be

$$Y_1 = pe_1 + n_1 w_1 - ps$$
$$Y_2 = pe_2 + n_2 w_2 + ps(1+r)$$

where "p" represents the market price of the single consumption good. Define "L" to be the length of each time period; normalizing both "p" and "L" to be equal to 1, we can define the individual's "full-income" levels to be

$$y_1 = e_1 + w_1 - s$$
$$y_2 = e_2 + w_2 + s(1+r)$$

which essentially represent potential income to the individual in each period and which will be used throughout the subsequent analysis. The effective intertemporal budget

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[2] Note the ambiguous role that wages play in determining indirect utility, in that higher wages simultaneously raise income but also raise the opportunity cost (or "price") of leisure. This ambiguity produces interesting precautionary saving implications when an individual faces wage income risk.

constraints show that perfect capital markets are assumed to exist as the individual is free to borrow and lend resources in period one at the same interest rate r:

$$c_1 \leq e_1 + n_1 w_1 - s$$
$$c_2 \leq e_2 + n_2 w_2 + s(1+r)$$

which may also be expressed as

$$c_1 + w_1(1-n_1) \leq y_1$$
$$c_2 + w_2(1-n_2) \leq y_2$$

where y_1 and y_2 represent full-income. These constraints are used to derive the indirect utility function in the "certainty" section of appendix A.

The individual's time and risk preferences are explicitly separated into two functionals:

(a) "time" preferences are captured by an isoelastic homogeneous temporal aggregator function (which is essentially an intertemporal utility index):

$$U(v_1(y_1, w_1), v^e(v_2(y_2, w_2))) = -\frac{1}{\delta_1}[(v_1(y_1, w_1))^{-\delta_1} + \beta(v^e(v_2(y_2, w_2)))^{-\delta_1}]$$

where $v_1(y_1, w_1)$ represents the level of first-period indirect utility, $v^e(v_2(y_2, w_2))$ represents the "certainty equivalent" level of second-period indirect utility, "β" represents the second-period discount factor, and "δ_1" represents a "time-preference" parameter which determines the desire for income-smoothing over time. Defining "η" as the elasticity of intertemporal substitution (EIS) between first-period indirect utility and second-period certainty-equivalent indirect utility, simple calculation reveals that

$$\eta = \frac{1}{\delta_1 + 1}$$

where $-1 \leq \delta_1 \leq +\infty$. Thus "$\delta_1$" is a parameter which is inversely related to the elasticity of intertemporal substitution, which measures the degree to which current and future utility are viewed as complementary by the individual. Graphically, the EIS parameter governs the curvature of the intertemporal utility index, with low EIS values generating greater concavity in the index than larger values. Since the index is assumed isoelastic its

curvature will be uniform, implying that the degree of intertemporal complementarity of utility for the individual will be a constant for any utility time-profile chosen. As we will see, a high degree of complementarity ($\eta < 1$) implies that the individual engages in precautionary saving and labor supply adjustments in an effort to smooth her income time-profile when future income sources are stochastic.

The time-additive formulation assumes the individual subjectively calculates a certainty equivalent level of period 2 indirect utility given all known income parameters, subjective probability distributions over any stochastic income parameters, and risk preferences. This certainty equivalent level of indirect utility is then plugged into the temporal aggregator function $U(\cdot)$, which is then maximized with respect to savings (the only remaining choice variable in period 1 given the form of the indirect utility function) to derive an optimal saving rule as a function of the income and preference parameters (see appendix B for examples). The isoelastic form of the aggregator ensures that nonpositive levels of utility in either period will not occur (Weil (1993).

(b) "risk" preferences are captured by a negative exponential utility function of the form

$$V(v_2(y_2, w_2)) = -\exp(-a(v_2(y_2, w_2)))$$

where "a" represents the coefficient of absolute risk aversion (CARA) and determines the concavity over states of the period (atemporal) utility index (which, like the intertemporal index above, will be constant). This specification assigns constant absolute risk aversion preferences to the individual (Ingersoll (1987), Hirshleifer and Riley(1994)), and turns out to be a particularly convenient representation for "removing" uncertainty from the model using Taylor's Theorem since risk attitudes will not change as the level of intertemporal wealth changes by construction. Like the direct utility function, opportunities arise for extending the current model by recasting risk preferences in another way (e.g. constant relative risk utility) for comparative purposes, but the task of solving out for any uncertainty in such a model would almost certainly be more difficult.

Solution Methods

For the case of stochastic second-period income (and thus stochastic second-period indirect utility) the following relationship will hold in period 1 before the uncertainty is resolved

$$V(v^e(\tilde{v}_2(y_2, w_2))) = E[V(\tilde{v}_2(y_2, w_2))]$$

or equivalently

$$v^e(\tilde{v}_2(y_2, w_2)) = V^{-1}(E[V(\tilde{v}_2(y_2, w_2))])$$

where "E" represents the expectations operator and a tilde denotes a stochastic variable. This relationship defines what is meant by a "certainty equivalent" level of indirect utility - the key is to obtain the expectation of the second-period indirect utility function which will generally be a nonlinear function of a stochastic income parameter for any reasonable specification of risk preferences.

Due to the nonlinearity of the problem, both within the risk preference specification and, in the case of stochastic second-period wages, within the second-period indirect utility function itself, extensive use of Taylor's Theorem occurs in the calculations to permit the derivation of closed-form optimal saving rules. Using second-order Taylor approximations of the term

$$V(\tilde{v}_2(y_2, w_2))$$

allow the expectations of the quadratic approximations to be expressed in terms of the first two moments of the distribution of any stochastic variables within the second-period indirect utility function (see appendix A for examples). This solution method is consistent with the assumption of how the individual forms subjective probability distributions as outlined above. Furthermore, because the period utility index will be approximated in this way with a quadratic functional, it will be entirely appropriate in the analysis to refer to a change in the variance of any stochastic income source as a change in income "risk".

Throughout the chapter, the second-period income sources are assumed to have the following moments when stochastic:

$$
\begin{bmatrix} \tilde{e}_2 \\ \tilde{w}_2 \\ (1+\tilde{r}) \end{bmatrix} \sim \left(\begin{bmatrix} \mu_e \\ \mu_w \\ (1+\mu_r) \end{bmatrix}, \begin{bmatrix} \sigma_e^2 & \rho_{ew}\sigma_e\sigma_w & \rho_{er}\sigma_e\sigma_r \\ \rho_{ew}\sigma_e\sigma_w & \sigma_w^2 & \rho_{wr}\sigma_w\sigma_r \\ \rho_{er}\sigma_e\sigma_r & \rho_{wr}\sigma_w\sigma_r & \sigma_r^2 \end{bmatrix} \right)
$$

where the relation $\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y$ for any random variables x and y is used. The use of Taylor's Theorem to deal with stochastic income in the model makes it unnecessary to assume that the individual assigns specific distributions to each income variable.[3]

The approximation of the certainty equivalent level of indirect utility in period two is then inserted into the aggregator function $U(.,.)$, which is then maximized with respect to first period saving to generate an optimal first order condition that can be implicitly differentiated to show how optimal saving responds to changes in the model parameters, including the level of perceived income risk. Several interesting results regarding factor allocations over time emerge in the model which are directly comparable to results derived using a standard expected utility framework in chapter four.

We consider six cases of income risk as well as a certainty case in the remainder of the chapter. The mathematical results shown here in summary form are explicitly derived in appendix A. Computer simulation results and graphical interpretations of them appear in the text.

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[3] However, the use of a negative exponential utility function to capture risk attitude in the model makes it particularly attractive to use normal distributional assumptions, since with an exponential moment-generating function a normal distribution allows for exact (rather than approximate) certainty-equivalent calculations. Negative utility outcomes could easily be ruled out by either assuming small income variances relative to means or by imposing a "bankruptcy" condition that prevents utility from falling below a certain level.

Certainty Case

When second-period income sources are all deterministic, it can be shown that the reduced-form first order condition for optimal period 1 saving (see appendix A) is

$$\left(\frac{v_1}{v_2}\right)^{-(\delta_1+1)} = \beta \ (1+r)\left(\frac{w_2}{w_1}\right)^{b-1}$$

Thus the MRS between first- and second-period indirect utility must be equal to the intertemporal price of transferring resources in equilibrium, where the intertemporal price is the product of the subjective discount parameter β, the gross rate of return on saving, and the ratio of real wages over the two time periods. A rise in either the discounting factor or the capital return rate will result in a greater willingness to substitute more utility in period 2 for less utility in period 1. In addition, a rise in period 1 wages relative to period 2 wages will also increase the substitution motive, whereas an increase in consumption preferences (i.e. an increase in b) will reduce the willingness to substitute utility forward in time and increase consumption and labor supply in both periods (see Marshallian demands).

Note that the relative value of the time endowments in the two periods, as captured by the intertemporal real wage ratio, contributes to the intertemporal price along with the term $\beta(1+r)$ which is the usual intertemporal price found in the literature. Thus, a rising (falling) wage schedule over time implies that the individual saves less (more) than the benchmark case where wages are assumed to be identical across time periods.

Implicitly differentiating the first order condition (see appendix A) shows that

$$\frac{\partial s}{\partial e_1} > 0, \ \frac{\partial s}{\partial e_2} < 0, \ \frac{\partial s}{\partial w_1} > 0, \ \frac{\partial s}{\partial w_2} < 0, \ \frac{\partial s}{\partial r} \ ?$$

assuming in all cases that $\delta_1 > -1$ (i.e. the EIS $< +\infty$). In all but one case these transitory income shocks stimulate a predictable pattern of saving responses that would be absent if the shocks were permanent. To the extent that current and future utility are viewed as

complementary the individual will attempt to smooth out any temporary income windfall gains and losses across both time periods. In particular, assuming that $0 < b < 1$ (i.e. both consumption and leisure have value to the individual), a fraction of any windfall gain to either first- or second-period endowment income or wages will, via saving/dissaving, be used to simultaneously raise consumption levels and to reduce labor supply in both periods; similarly, if $0 < b < 1$, a fraction of any windfall loss to either first- or second-period endowment income or wages will, via saving/dissaving, be spread out over both time periods and result in lower consumption and higher labor supply in each period.

Saving responses to endowment income changes are dampened by higher interest returns on saving but are unaffected by time preference. Saving responses to wage changes in period 1 and 2 are also dampened by higher saving returns, whereas changes in time preferences are relevant here in that a rise (fall) in δ_1 will enhance (reduce) the saving response as the individual becomes less (more) willing to witness fluctuations in expected income and utility across time periods.

The sign of $\partial s/\partial r$, where

$$\frac{\partial s}{\partial r} = \frac{\beta \left(\frac{w_2}{w_1}\right)^{b-1} - (\delta_1+1)sAw_2^{b-1}(v_1)^{-(\delta_1+1)}(v_2)^{\delta_1}}{(\delta_1+1)A(v_1)^{-(\delta_1+2)}(v_2)^{\delta_1}\left(v_2w_1^{b-1}+(1+r)v_1w_2^{b-1}\right)}$$

is ambiguous as expected, reflecting the opposing pull of the substitution and income effects of factor price changes on factor supplies. The substitution effect is primarily a function of the subjective discount factor and the relative wage rate, and a rise in either will augment the substitution effect and drive the level of saving up. The income effect is driven by the EIS parameter δ_1. As δ_1 rises (falls) the income effect of a change in capital returns becomes stronger (weaker) which results in a lower (higher) level of saving. The sign of the expression depends critically on δ_1. If the desire to smooth income and utility

over time becomes strong enough a higher interest rate will actually drive the net level of saving down.[4]

A high current level of saving also strengthens the income effect of a change in interest returns, whereas low levels of current saving encourages greater saving when interest returns rise.

Stochastic Endowment Income

When second period endowment income is random (e.g. uncertainty over future Social Security and Medicare benefits), it is shown in Appendix A that the reduced-form first order condition from the basic model is

$$\left(\frac{v_1}{v^e(\tilde{v}_2)}\right)^{-(\delta_1+1)} = \beta\,(1+r)\left(\frac{w_2}{w_1}\right)^{b-1}$$

Thus the equilibrium MRS between first-period indirect utility and second-period certainty equivalent indirect utility when an exogenous income source is stochastic will be identical to the equilibrium MRS between first- and second-period indirect utility in the certainty case since the intertemporal prices in each case are themselves equal and exogenous. This identity follows since endowment income is assumed exogenous in the second period (unlike labor and capital income), and thus changes in the variance of such an income source cannot affect the rate at which the individual is able to transform resources from the present to the future. The ability to transform wealth in this way is determined by the saving rate and relative wage rate.

Implicitly differentiating the first order condition (see appendix A) shows that

$$\frac{\partial s}{\partial \sigma_e^2} = \frac{aAv_1(w_2^{b-1})^2}{2(v^e(\tilde{v}_2)w_1^{b-1}) + (1+r)v_1 w_2^{b-1}} > 0 \quad \forall\, a > 0$$

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[4] This indirectly suggests that any evidence of income-smoothing behavior in an environment of unstable capital returns is a potentially useful measure of the strength of the income effect of such volatility over saving behavior.

Thus there exists only an income effect on saving of a change in the variance of second-period endowment income - a rise in the variance boosts first-period precautionary saving and a fall reduces precautionary saving. Note that in the certainty case precautionary saving (or precautionary dissaving) did not occur because there was an absence of future income uncertainty against which to insure. Here second-period income is stochastic, which introduces the notion of precautionary saving as a form of ex ante insurance against low outcomes of the random income variable.

The interesting feature of the expression is that neither the sign nor the magnitude of the saving response depends on the δ_1 parameter, as demonstrated by the simulation results appearing in Figures 3.1 and 3.2.[5,6] The graphed saving functions here, and in all six stochastic cases considered in the chapter, are in general linear since all variance and covariance terms enter the second-period indirect utility functions and intertemporal price equations linearly. As Figures 3.1 and 3.2 show, a rise in endowment income risk generates a pure income effect on saving of significant magnitude and results in greater precautionary saving from a risk averse individual. A lower EIS raises the absolute level of saving slightly, but does not affect the impact of a change in endowment income risk on

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[5] Computer simulations were conducted for all six stochastic cases discussed in the chapter. Nonlinear maximization of the objective functions produced optimal saving responses to future income risk under a variety of different model parameterizations, some of which are presented here in graphical form. Throughout the simulations several parameters were held constant. These fixed assignments include: $b = 0.7$ (except where noted), $\beta = 0.95$, $e_1 = 1$, and $r = 0.01$. In addition, the values assigned to the variance terms in the multivariate cases are as follows: $\sigma_e^2 = 0.01$, $\sigma_r^2 = 0.01$, and $\sigma_w^2 = 0.02$. These assignments are not meant to be actual microeconomic calibrations of the model but rather serve to highlight the behavioral implications of income risk on factor allocations over time. These values represent a slight modification of those used by Werner (1990) - a lower β value is chosen to expand the relevant time-horizon, and additional variance assumptions regarding endowment and wage income are made which do not appear unreasonable. Practice simulations show the relative insensitivity of the basic results to minor changes in the following parameters: r, e_1, and all variance and covariance terms.

[6] The computer simulations consider two values of the time preference parameter δ_1 : 0.43 and 9. These values correspond to the EIS values 0.7 and 0.1, respectively, which are meant to represent "high" and "low" tolerance for intertemporal substitution of utility. The absolute risk aversion parameter a assumes four possible values in the simulations: 0, 2, 5, and 8. To generate both "high" and "low" saving levels, the second period endowment parameter e_2 could assume values of 0, 0.1, and 0.9. As expected from the mathematical results in this chapter, these different assignments affect the level of first-period saving but do not, by themselves, affect the marginal saving implications. Finally, to present a clear picture of precautionary saving responses to various income risks, period 1 wages and (mean) period 2 wages are assumed identical and can assume "low" (0.2), "moderate" (1.0), and "high" (2.0) values relative to first-period endowment income.
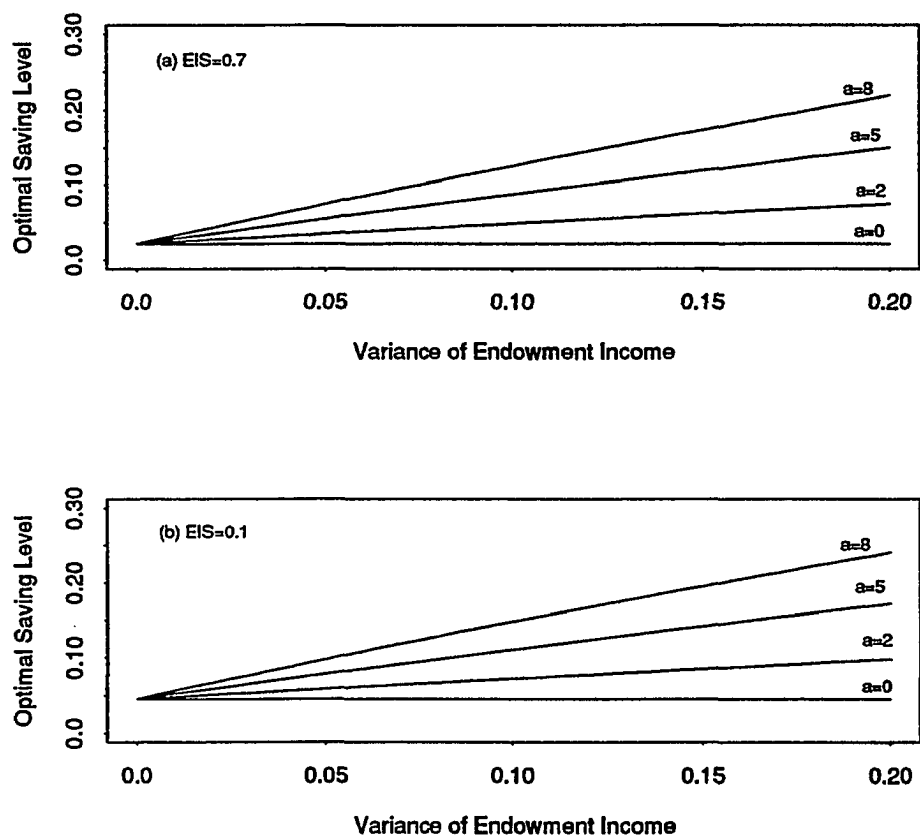
Figure 3.1. Optimal saving path with moderate wages and $e_2 = 0.9$.
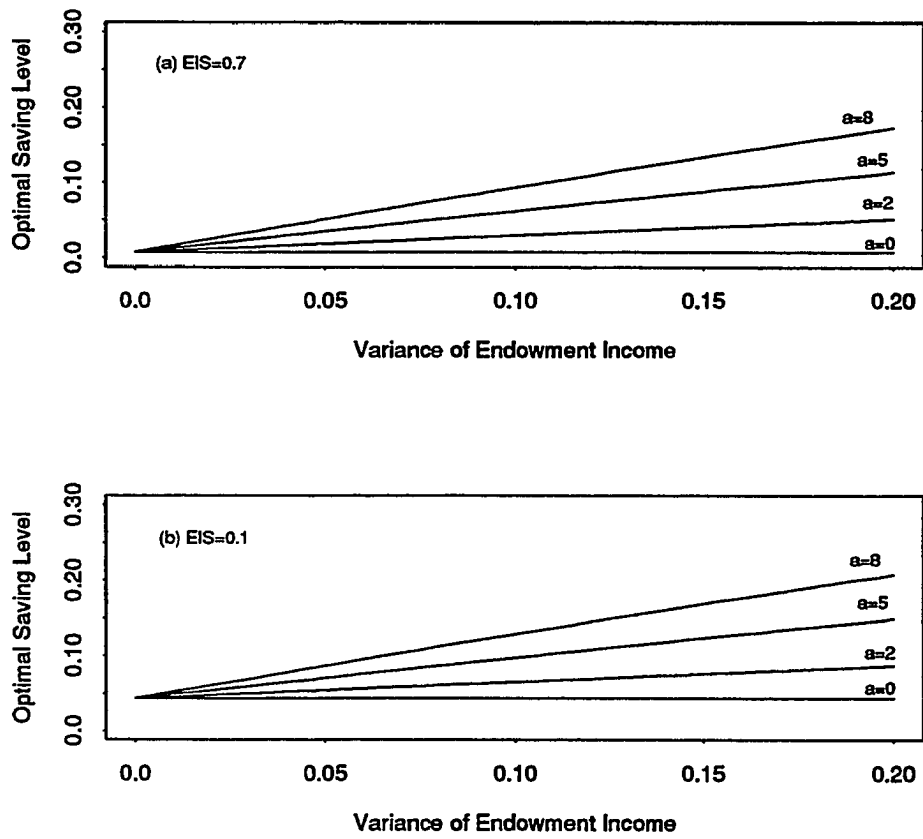
Figure 3.2. Optimal saving path with high wages and $e_2 = 0.9$.

marginal saving since the slope of the saving function is unaffected by it. Higher risk aversion clearly enhances the income effect of a rise in exogenous income risk on precautionary saving, while higher period 2 wages dampens the effect. The latter result occurs since higher future wages make ex post labor supply a more attractive way to hedge against income risk than ex ante saving - as a result, saving levels drop when higher wages are anticipated in the future.

Thus, assuming that $\delta_1 > -1$ (a value equal to -1 would induce a corner solution regarding the saving of first-period income) the individual's EIS preference parameter is completely irrelevant to determine either the qualitative response of precautionary saving to endowment income risk or whether optimal saving will be higher or lower compared to the certainty case.

The crucial preference parameter here is the risk attitude measure a, which is primarily responsible for the sign of the expression. If a > 0, the individual is risk averse and precautionary saving will occur if $\sigma_e^2 > 0$; if a = 0, the individual is risk neutral and precautionary saving will not occur. Furthermore, greater risk aversion will tend to amplify the level of precautionary saving that already exists, whereas lower aversion to risk will push down the current level of precautionary saving. Here, in contrast to cases such as stochastic wage and capital income, examined later, which affect the intertemporal price of wealth transfers, (i) risk attitude alone drives precautionary saving behavior, and (ii) risk attitude drives the income effect of a change in future income risk on saving behavior.

The influence of risk attitude on saving behavior is straightforward. Assuming a risk averse individual exists, this individual will interpret a rise in the variance of second-period endowment income as a reduction in the certainty equivalent level of second-period income and utility, and will attempt to compensate for this reduction by raising the level of precautionary saving in period 1 (again assuming that first- and second-period utility are not viewed as perfect substitutes, implying that $\delta_1 > -1$). Equivalently, as the level of risk aversion itself rises for a given variance in second-period endowment income, a

corresponding reduction in the certainty equivalent level of second-period income and utility occurs and, again, a boost in precautionary saving will result in period 1.

The absence of the intertemporal substitution parameter is less straightforward, and results from the form chosen for the intertemporal utility index. A change in the level of certainty equivalent endowment income in period 2, unlike a change in the certainty equivalent wage rate or capital return rate which affects the slope of the intertemporal price surface, will result in a parallel shift of the certainty equivalent intertemporal wealth constraint, enhancing or reducing income and utility prospects in each period proportionately. Given that the intertemporal utility index is homogeneous (and thus homothetic), the resulting intertemporal income expansion path will be linear and the "demand" for utility will be proportionate across the two periods as intertemporal wealth rises and falls. Thus, the level of precautionary saving will be a constant proportion of the level of perceived wealth regardless of the curvature of the intertemporal utility index. Risk attitudes, by contrast, play a crucial role here by determining the magnitude of the shift in the certainty equivalent intertemporal wealth constraint following a change in the variance of second-period endowment income, with higher (lower) risk aversion amplifying (dampening) the shift in the constraint and thus the change in precautionary saving observed.

An increase in the relative wage ratio (w_1/w_2) will enhance the precautionary saving motive by flattening the wealth constraint and thus raising the income expansion path. Economically, a higher wage ratio not only encourages greater labor supply and enhances income possibilities in period 1 relative to period 2 but also makes it less attractive to a risk averse individual to use labor supply in period 2 to engage in a form of ex post insurance against negative shocks to second-period endowment income, all of which encourage the use of precautionary saving as ex ante insurance against low future income. For a risk averse individual this is equivalent to "making hay while the sun shines" (van der Ploeg (1993)).

Examination of this theme of alternative income insurance mechanisms to buffer against future income risk was a primary reason for the construction of the intertemporal model used here, and it is a theme which will appear often in the proceeding sections of this chapter. There exists only two ways in this model for the individual to insure herself against low levels of second-period income: (i) ex ante precautionary saving in period 1, and (ii) ex post reactionary labor supply in period 2. Restrictions on one form of insurance will induce distortions on the other form, in that labor supply barriers (e.g. lower relative real wages in period 2 offered due to falling aggregate demand) will augment precautionary saving motives, whereas capital barriers (e.g. lower real returns offered due to government capital controls) augment labor supply decisions.

Income insurance will not take place unless current and future utility levels are viewed to some small degree as complementary. If more alternative income insurance mechanisms became available to the individual, we could be sure that different (probably smaller) saving and labor supply responses would be observed in the model when income risk is introduced than occur here.

A higher capital return dampens the precautionary saving response here for obvious reasons - a higher capital return requires less saving to boost the level of second-period certainty equivalent income a given amount. This result is in conformity with the pure income effect observed here on saving behavior as the normal substitution effect of a change in capital returns on saving behavior is not evident in the comparative statics above.

Finally, as the preference for leisure strengthens (i.e. as b falls) in both periods, the precautionary saving motive will also strengthen. Again, as in the case of a rising first-period wage relative to second-period wages, greater preference for leisure (e.g. by an individual planning an early retirement) will make second-period labor supply a relatively unattractive way to hedge against low second-period income compared to an increase in labor supply and precautionary saving in period 1. This will be a consistent behavioral observation across the entire spectrum of stochastic income sources examined.

Stochastic Capital Income

A particularly interesting case exists when the gross real interest rate is random (e.g. during periods of high, unpredictable inflation or other financial market instability), where it is shown in Appendix A that the reduced-form first order condition from the basic model is

$$\left(\frac{v_1}{v^e(\tilde{v}_2)}\right)^{-(\delta_1+1)} = \beta \left(\frac{w_2}{w_1}\right)^{b-1} \left((1+\mu_r) - asAw_2^{b-1}\sigma_r^2\right)$$

Thus the equilibrium MRS between first-period indirect utility and second-period certainty equivalent indirect utility will be equal to the product of the subjective discount factor, the ratio of real wages over the two periods, and now a risk-adjusted gross rate of return on the level of period 1 saving. Note that the risk-adjusted rate falls if (i) risk aversion rises, (ii) period 2 real wages fall, (iii) consumption preferences rise (which will raise labor supply and precautionary saving in period 1), (iv) saving itself rises, and (v) the variance of saving returns rises. All five of these events will reduce the risk-adjusted rate of capital return by increasing the covariance between second-period indirect utility and capital returns, which is represented by the term $Aw_2^{b-1}s\sigma_r^2$ in the first order condition (Werner, p. 35). Thus, the risk-adjusted level of capital return in the first order condition can be represented as

$$E(1+\tilde{r}) - a\,Cov\left((1+\tilde{r}), \tilde{v}_2\left(c_2, (1-n_2)\right)\right)$$

A reduction in the risk-adjusted return will, according to the first order condition, make the individual less willing to substitute greater utility in period 2 for lower utility in period 1. The opposing forces of the substitution and income effects of capital return risk on saving behavior are apparent by observing that the first order condition implies that greater saving by the risk-averse individual due to a reduction in certainty-equivalent second-period income will simultaneously reduce her incentive to save by lowering the risk-adjusted capital return.

Implicitly differentiating the first order condition shows that

$$\frac{\partial s}{\partial \sigma_r^2} = \frac{C_1\left(\left(\delta_1 + 1\right)C_3 - C_2\right)}{U_1}$$

where C_1, C_2, C_3, and U_1 are all positive for reasonable specifications of the model parameters, where "reasonable" means (i) $0 < b < 1$ and (ii) $a \geq 0$ (see appendix A). Thus the sign of the expression, and thus the slope of the saving function, depends critically upon the magnitude of the δ_1 parameter which determines both the degree to which utility is viewed as complementary over time by the individual and the importance of the income effect on the precautionary saving motive. When δ_1 is low the individual views first- and second-period utility as substitutes and will be relatively more willing to watch utility fluctuate freely between the two periods, implying that the income effect on precautionary saving of the reduction in risk- adjusted second-period income resulting from an increase in the variance of capital returns will be small. The substitution effect of the presence of capital income risk will then likely control saving behavior, causing first-period precautionary dissaving to occur and producing a negatively-sloped saving function. This dissaving behavior is indicated by observing that $\partial s/\partial \sigma_r^2$ will be negative at low levels of δ_1.

In contrast, a high level of δ_1, where the individual views first- and second-period utility as complementary, will magnify the income effect on precautionary saving by implying that the individual is relatively less tolerant of intertemporal income and utility fluctuations and prefers income and utility smoothing by varying ex ante precautionary saving and/or ex post labor supply. At high enough levels of δ_1 (which is unbounded from above) it is conceivable that the income effect on saving of introducing stochastic capital returns will overpower the substitution effect, with the net result that saving will actually increase above the certainty level when returns are made risky and producing a positively-sloped saving function.

Assuming risk averse behavior (a > 0) ensures that δ_1 will determine the direction of the saving response to changes in the variance of capital returns; however, risk aversion is necessary as well as sufficient to obtain this result for reasonable model parameters. It is clear that risk aversion serves to amplify both the motive for precautionary saving and dissaving since it influences both the substitution and income effects of income risk on saving behavior. It is also clear that if the individual were risk neutral (a = 0), then

$$C_1 \equiv asA(w_2^{b-1})^2$$

will equal zero and saving will be unresponsive to changes in the variance of capital returns; risk loving behavior (a < 0) will reverse the findings of the preceding paragraph.

Thus Selden's (1979, p. 80) claim that the risk aversion parameter is, contrary to intertemporal isoelastic EU, irrelevant to determine (i) the qualitative effect on saving of a change in capital income risk, and (ii) whether optimal saving is larger in the presence or absence of capital income risk, is shown to be susceptible to misinterpretation - risk preference matters in both cases unless they are a priori restricted (unlike the claims made earlier about the role of the EIS preference parameter when endowment income is stochastic).

In any case, the magnitude of the saving response will clearly depend on both risk attitude and time preference in this model, as Figures 3.3 and 3.4 demonstrate. The graphs show evidence of both a negative substitution and positive income effect at work on saving behavior when capital returns are stochastic. At low wage levels, which makes ex ante saving an attractive way to insure against low income realizations in period 2, higher capital return risk produces an initial positive income effect on saving when the EIS is low. However, as the level of risk rises the negative substitution effect begins to overpower the positive income effect, and eventually the level of risk rises so high that saving falls below the certainty level. At higher EIS values the income effect disappears altogether along with the desire to smooth income over the two periods. Interestingly, while the level of risk aversion enhances both the income and substitution effects, the direction of the saving
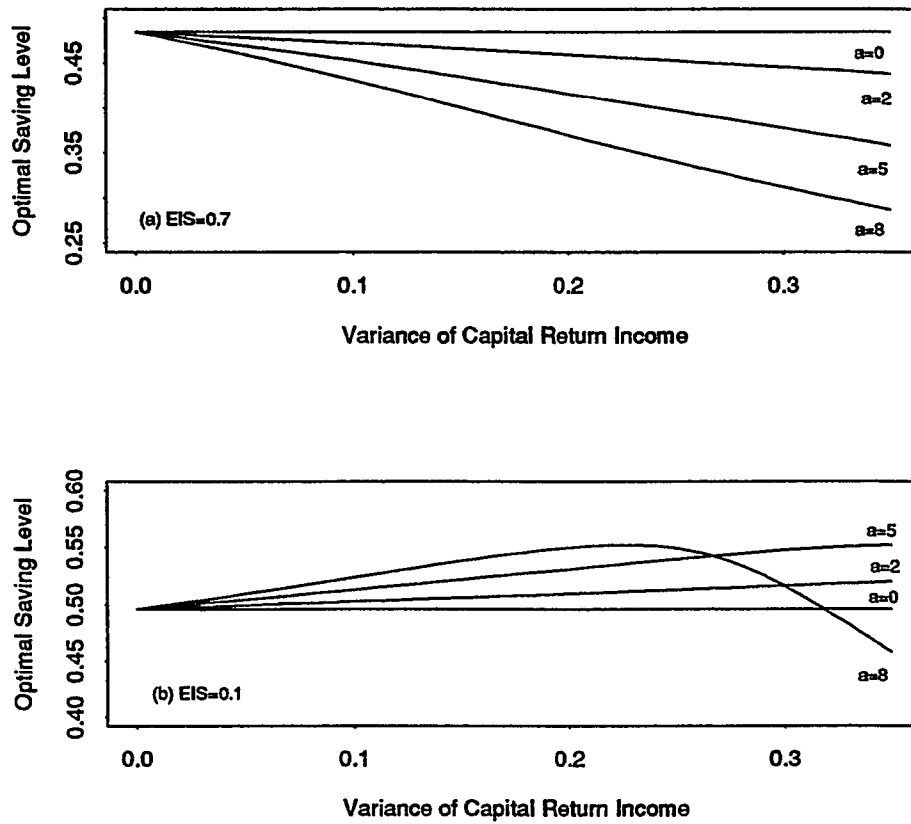
Figure 3.3. Optimal saving path with low wages and $e_2 = 0$.
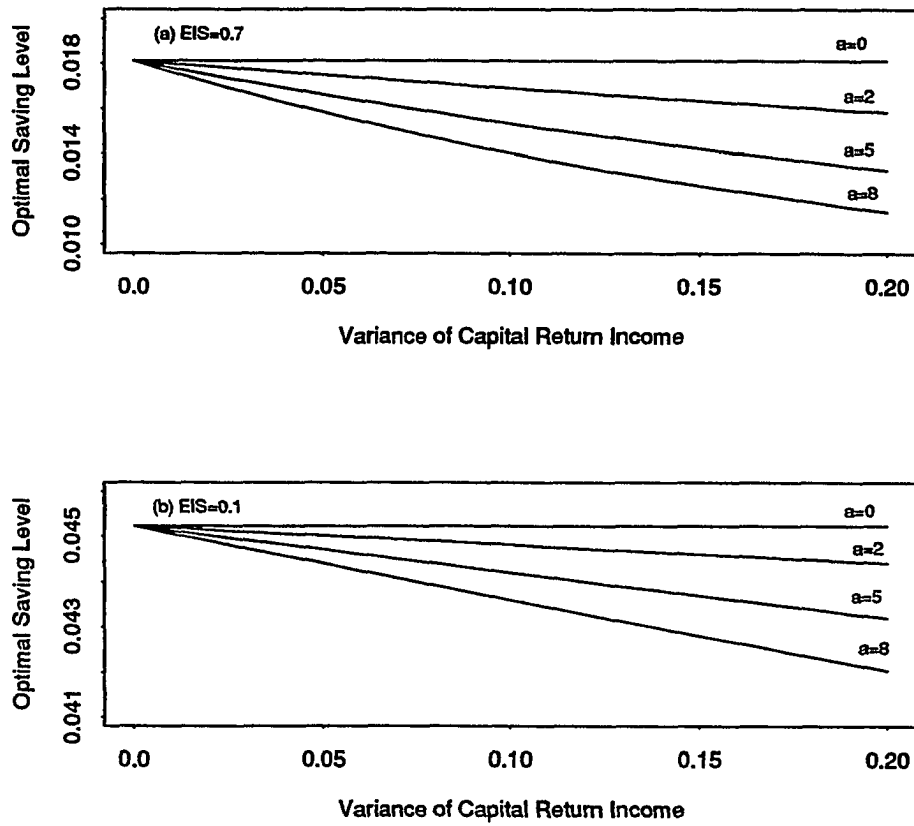
Figure 3.4. Optimal saving path with moderate wages and $e_2 = 0.9$.

response depends critically on the EIS specification rather than risk attitude. At higher wage levels the substitution effect dominates regardless of the EIS or risk aversion parameters (assuming a > 0) - again, ex post labor supply serves as a better way to hedge risk than ex ante saving, with the result that higher risk drives saving lower (and drives expected second-period labor supply higher).[7] Unsurprisingly, a rise in risk aversion enhances the precautionary dissaving motive when risk increases.

In addition, saving responses to changes in capital return risk will occur even when current and future utility are viewed as perfect substitutes (i.e. $\delta_1 = -1$). A pure substitution effect will lower (raise) the level of precautionary dissaving in response to lower (higher) risk. This stands in contrast to results using models which incorporate standard state- and time-separable isoelastic von Neumann-Morgenstern EU preferences which suggest that precautionary saving should be completely unresponsive to capital risk when current and future utility are viewed indifferently (i.e. the watershed logarithmic risk preference case; see Samuelson (1969), Merton (1969), Sandmo (1970), Rothschild and Stiglitz (1971), Mirrlees (1974), and Sandmo (1974)) and should increase above the certainty level only if the coefficient of relative risk aversion is greater that one (i.e. the atemporal utility index is more concave than the log-additive case).

In contrast to the case of stochastic endowment income, where the risk aversion parameter determines the size of the income effect of income risk on precautionary saving behavior, stochastic capital income produces the result that the risk aversion parameter determines the size of the substitution effect, whereas both the risk aversion and EIS parameter jointly determine the size of the income effect, which is intuitively appealing.

The income effect on saving is reinforced by (i) an increase in second period relative wages (w_2/w_1), which reduces the saving adjustment necessary in period 1 to compensate for a given change in capital income risk since mean second-period income

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[7] Note the change in the vertical axis scaling between Figures 3.4(a) and 3.4(b) - the increase in wages clearly highlights the important role that the EIS parameter plays in motivating precautionary saving behavior.

will be larger relative to the variance caused in second-period income by such risk - this effect underscores the role that labor supply fulfills as a source of ex post insurance against income risk; and (ii) a reduction in the fraction of income derived from saving.

As preferences for leisure rise (i.e. as b falls), C_1 falls and will overpower the rise in C_2 assuming that great intertemporal wage differences do not exist. This implies a boost for the income effect of capital return risk on precautionary saving by reducing the attractiveness to the individual of using labor supply in the second-period as an income buffer - ex ante insurance becomes preferred as preferences for leisure rise.

Stochastic Wage Income

When second-period real wage income is random it is shown in the appendix that the reduced-form first order condition is

$$\left(\frac{v_1}{v^e(\tilde{v}_2)}\right)^{-(\delta_1+1)} = \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} (1+r)\left(1+K_w \sigma_w^2\right)$$

where K_w is a complex function of the model parameters (see appendix A) which will be positive for reasonable values of its arguments. Thus, we see here that, unlike the variances of either stochastic endowment income or capital returns, an increase in the variance of second-period wages will induce a greater willingness to substitute higher certainty equivalent second-period indirect utility for lower first-period indirect utility.

Implicitly differentiating the first order condition we see that

$$\frac{\partial s}{\partial \sigma_w^2} = \frac{C_4 + \left(\delta_1 + 1\right) C_5}{U_2}$$

where U_2 will be positive for reasonable specifications of the model parameters. The signs of C_4 and C_5 are ambiguous, which follows because of the ambiguous role wages play in

determining indirect utility by simultaneously affecting both income and the opportunity cost of leisure.

As the simulations show, C_4 is positive, C_5 is negative, and the sign of the entire expression is positive for the parameter settings used there. The income and substitution effects reinforce one another to drive up precautionary saving when the level of wage income risk rises. Of particular interest here is that with the parameter assignments used, a fall in the EIS enhances the absolute level of saving (in the absence of risk) but actually dampens the marginal saving response to an increase in risk given any level of risk aversion, as shown in Figures 3.5 and 3.6.[8] The reason for this counterintuitive result is straightforward: a risk averse individual with a high tolerance for intertemporal substitution will be more willing to take advantage of the fact that the certainty-equivalent return to labor falls relative to capital return when wage income risk rises by augmenting the current capital supply. Thus, as wage income risk rises, such an individual has two reasons to boost saving: (i) a precautionary response to greater income risk and (ii) a reallocation of factor supply in response to a certainty-equivalent factor price adjustment.

Thus, precautionary saving will be driven by the pure income effect of a change in the variance of second-period wages, and the expression above will be positive for a risk averse individual for reasonable values of the EIS preference parameter. However, both the EIS and coefficient of absolute risk aversion (CARA) parameters, as well as the expected intertemporal wage ratio, will affect the magnitude of the saving response, in that a lower EIS and higher CARA will stimulate greater precautionary saving when wage risk rises whereas higher EIS and lower CARA will reduce the level of precautionary saving.

Interestingly, the expression indicates that a higher capital return will stimulate precautionary saving when wage uncertainty exists in period 2. Economically, this can be explained since the second-period income risk can then be dealt with at a relatively lower certainty-equivalent intertemporal cost by ex ante saving instead of ex post labor supply

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[8] Again the importance of the EIS parameter is displayed in comparing the marginal saving responses to risk in Figures 3.5(a) and 3.5(b), as well as between Figures 3.6(a) and 3.6(b).
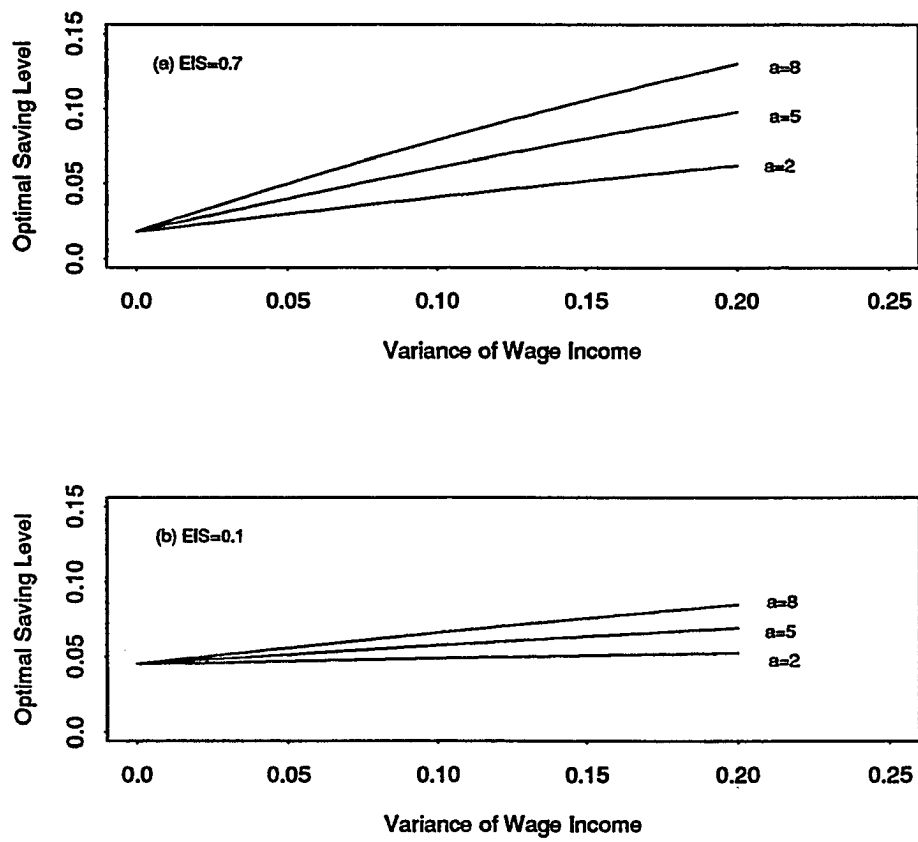
Figure 3.5. Optimal saving path with moderate wages and $e_2 = 0.9$.
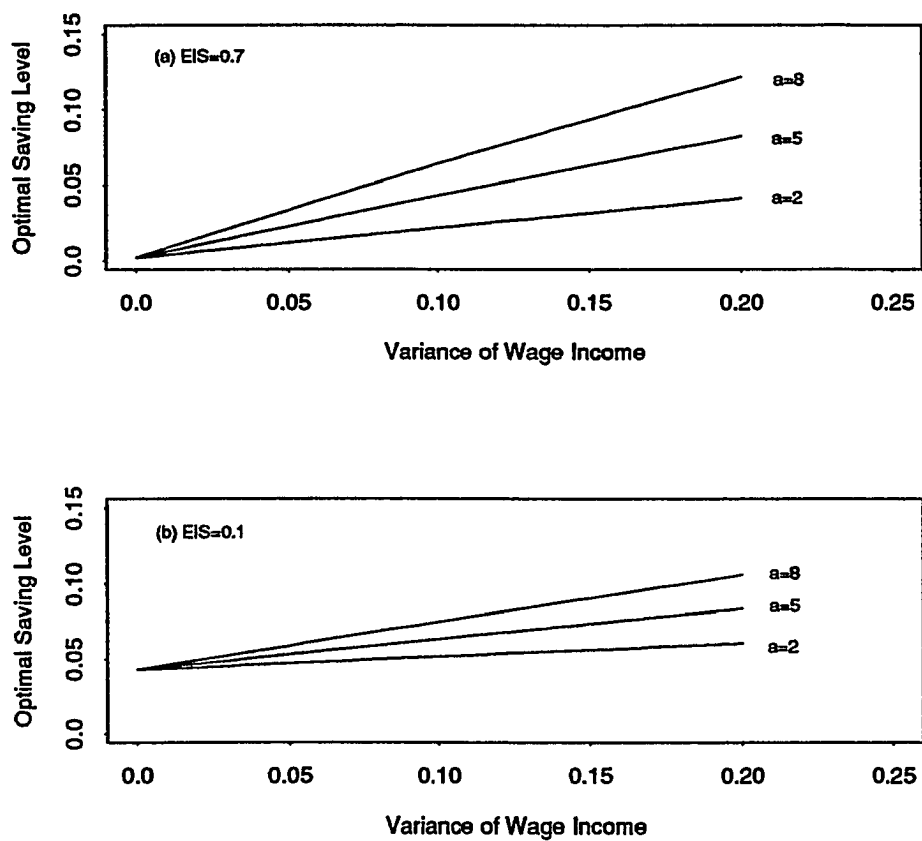
Figure 3.6. Optimal saving path with high wages and $e_2 = 0.9$.

adjustments when interest rates rise, especially with stochastic wages. Mathematically, a rise in saving returns reduces the intertemporal price of transferring resources from period 1 to period 2 as depicted in the right-hand side of the equality in the first order condition, which will raise both the slope of the certainty-equivalent intertemporal wealth constraint and the income expansion path, driving the level of precautionary saving higher.

High expected relative wages in period 2 will also push down the level of precautionary saving here. This follows logically since higher period 2 wages implies that ex post labor supply becomes a relatively more attractive income insurance option than ex ante precautionary saving to the individual. As a result, greater period 1 consumption and leisure increase in response to a rise in both expected mean wages and the expected variance of wage income in period 2.

Finally, as expected, the consumption preference parameter b is directly related to the level of precautionary saving. As Figure 3.7 demonstrate, precautionary saving can actually fall in response to higher wage risk if b assumes perversely low values. This saving response occurs because the reduction in the opportunity cost of leisure associated with higher second-period wage risk actually enhances the certainty-equivalent level of indirect utility in period 2 for a risk averse individual who greatly enjoys leisure. The increase in expected utility in period 2 will cause saving to fall if the individual is also highly averse to intertemporal substitution of utility.

Stochastic Endowment and Wage Income

When both second-period endowment and wage income are random, it can be shown that the reduced-form first order condition from the basic model is

$$\left(\frac{v_1}{v^e(\tilde{v}_2)}\right)^{-(\delta_1+1)} = \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} (1+r)\left(1+K_w\sigma_w^2 + K_{ew}\,\sigma_{ew}\right)$$
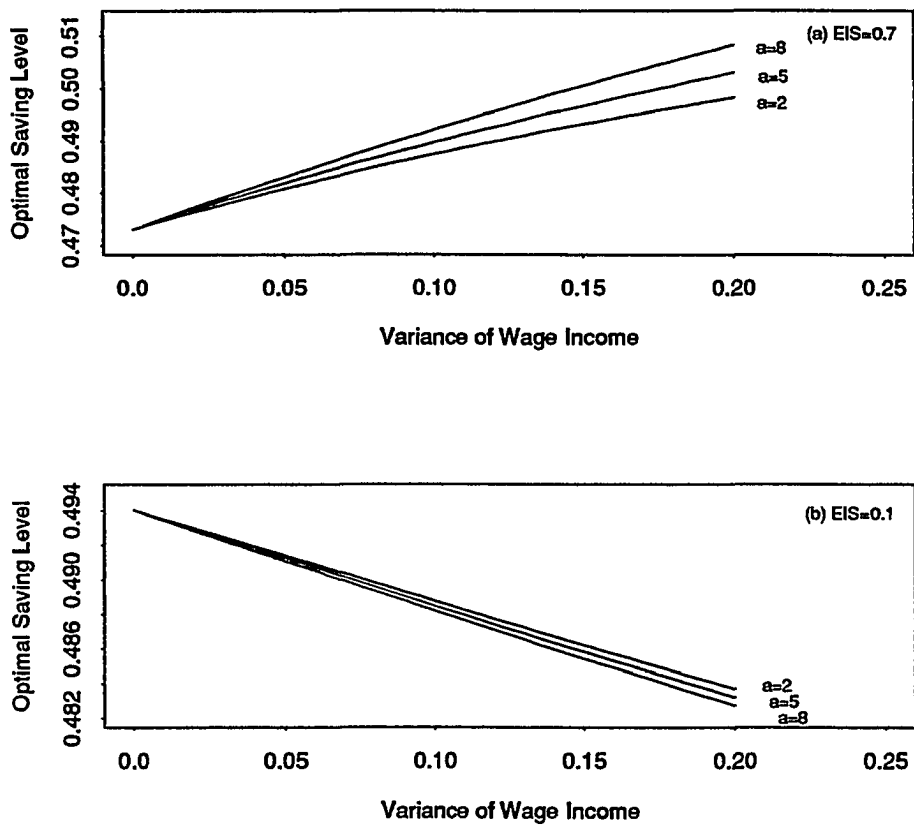
Figure 3.7. Optimal saving path with moderate wages, $b = 0.3$, and $e_2 = 0$.

where K_w has already been shown to be positive for reasonable parameter specifications. K_{ew} is a simple function of the model parameters which is also positive for reasonable values of its arguments. The interesting observations on this first order condition are that the covariance term, as well as the two variances, enter the intertemporal price expression, and do so linearly, which is an artifact of the way that uncertainty was removed through the linearization process of the Taylor approximations on the risk functional V(.), and that a higher covariance between the realizations of these two income sources, in addition to higher variances, will increase the incentives to substitute utility forward in time.

Implicitly differentiating the first order condition, and noting that $\sigma_{ew} = \rho_{ew}\sigma_e\sigma_w$, where ρ_{ew} is the correlation coefficient between endowment and wage income, we see that

$$\frac{\partial s}{\partial \rho_{ew}} = \frac{C_6 + (\delta_1 + 1)C_7}{U_3}$$

where C_6 and U_3 are positive, whereas C_7 is ambiguous, for reasonable specifications of the model parameters (see appendix A). As the simulation results in Figures 3.8 and 3.9 show, C_7 is negative for b = 0.3, as is the sign of the expression as a whole, for the parameter settings used there. A risk averse individual who enjoys leisure interprets a rise in income correlation as an increase in second-period indirect utility due to the associated decrease in the "price" of leisure, and responds by reducing saving in an attempt to smooth the utility time profile. Thus, assuming risk averse behavior is sufficient to sign the comparative static expression negative (for all $\delta_1 > -1$; i.e. assuming v_1 and v_2 are not perfect substitutes).[9]

Of particular interest here is the observation that correlation between stochastic wage and endowment income produces an effect on precautionary saving behavior that is independent of any changes in the variance of either income source. A perceived increase in income correlation is viewed by a risk averse individual as an increase in certainty-equivalent second-period indirect utility (via a corresponding reduction in the opportunity

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[9] Note that for b = 0.7, the income effect of greater correlation will dominate and the saving functions will all have a positive slope for a > 0.
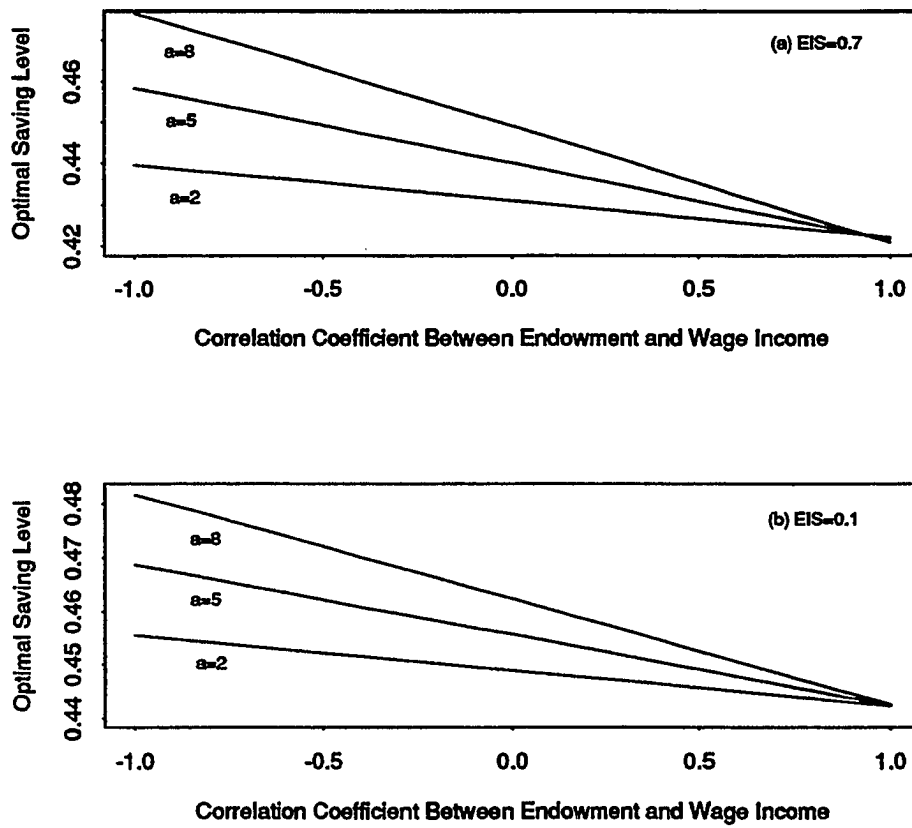
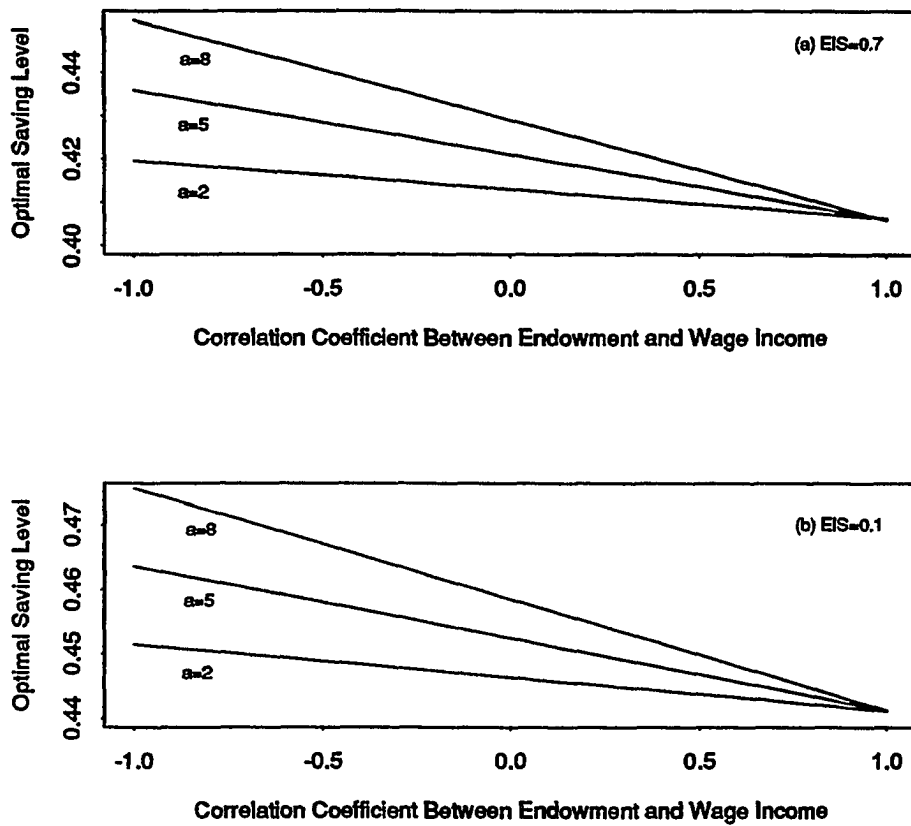Figure 3.8. Optimal saving path with moderate wages, $e_2 = 0.1$, and $b = 0.3$.

Figure 3.9. Optimal saving path with high wages, $e_2 = 0.1$, and $b = 0.3$.

cost of leisure), resulting in a fall in precautionary saving, whereas a perceived decrease in the correlation increases saving. These responses are strengthened by (i) an increase in risk aversion, (ii) a decrease in the EIS parameter (i.e. an increase in δ_1), (iii) a rise in capital returns, and (iv) an increase in leisure preferences, for reasons implied earlier in the chapter. A multivariate intertemporal optimization has thus revealed an additional motive behind precautionary saving - a perceived increase in the correlation between future income sources that separately may possess only small stochastic components.

Stochastic Wage and Capital Income

A different and perhaps more interesting case exists when both second-period wages and capital income are random. It can be shown that the reduced-form first order condition from the basic model is

$$\left(\frac{v_1}{v^e(\tilde{v}_2)}\right)^{-(\delta_1+1)} = \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} \left((1+\mu_r)+K_w\sigma_w^2+K_r\sigma_r^2+K_{wr}\sigma_{wr}\right)$$

where K_w, K_r, and K_{wr} are complex functions of the model parameters. As previously shown in the univariate cases, K_w will be positive and K_r will be negative for reasonable specifications of the model parameters. The sign of K_{wr} is ambiguous for reasonable values of its arguments and cannot easily be determined without resorting to simulation.[10]

Implicitly differentiating the first order condition, and noting that $\sigma_{wr} = \rho_{wr}\sigma_w\sigma_r$, where ρ_{wr} is the correlation coefficient between wage and capital return income, we see that

$$\frac{\partial s}{\partial \rho_{wr}} = \frac{C_8 + (\delta_1+1)C_9}{U_4}$$

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[10] This observation, as well as the following comparative static expression, stand in contrast to the results in chapter four, where an intertemporal expected utility model is used. There an easily-interpreted sign for K_{wr} results for the popular isoelastic case if one overlooks the inherent confusion of preferences.

where U_4 is positive while C_8 and C_9 are both ambiguous (see appendix A). The simulation results in Figures 3.10 and 3.11 show that either the substitution or the income effect of the change in income correlation on precautionary saving behavior can dominate, depending critically on the EIS parameter. At high wage levels we see a familiar pattern of a dominating substitution (income) effect when the EIS is high (low), with the risk aversion parameter serving to merely enhance whichever effect prevails. In addition, we see that while higher income correlation simply produces higher precautionary saving for individuals with a low tolerance for intertemporal substitution, such correlation produces more complex behavior in individuals with higher tolerance. While the substitution effect dominates saving behavior for these individuals, higher wages promote greater sensitivity to income correlation, in that for a risk averse, high-wage individual large positive (negative) income correlation will result in lower (higher) saving relative to the certainty case - this contrasts to the case of moderate wage income in Figure 3.10(a) in which a risk averse individual will always save more in the presence of income correlation than in the certainty case. The greater risk sensitivity of the high-wage individual is natural since a larger fraction of total second-period income is at risk for her relative to the moderate-wage individual.

Thus, we note that both an income and a substitution effect on saving behavior occur when these two income sources are perceived to be correlated. In particular, if expected capital income represents a large fraction of expected second-period income, a rise in the correlation between stochastic wage and capital return realizations will drive the level of precautionary saving up. This makes sense since the individual will be more sensitive to the risk characteristics of capital returns if capital income comprises a significant share of expected second-period total income and will be more motivated to insure herself against any low realizations of this income source. As either the share of expected capital income relative to expected second-period income falls or the EIS rises, the substitution effect of a change in the correlation term will become stronger relative to the income effect, and it becomes more likely for precautionary dissaving to occur when
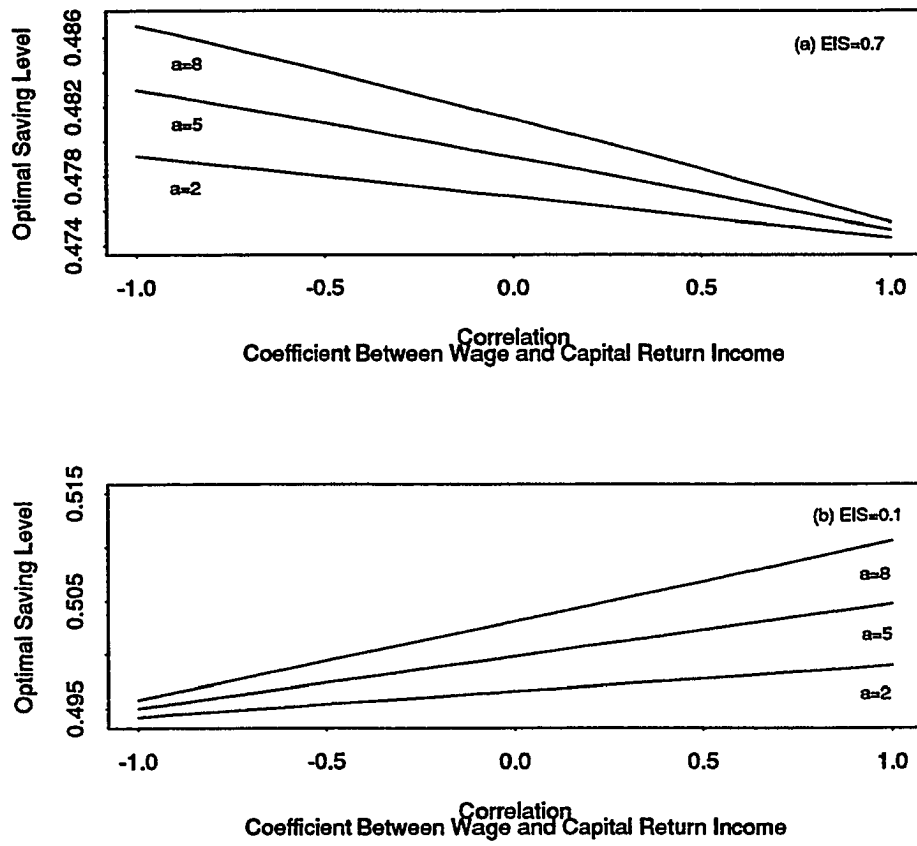
Figure 3.10. Optimal saving path with moderate wages and $e_2 = 0$.
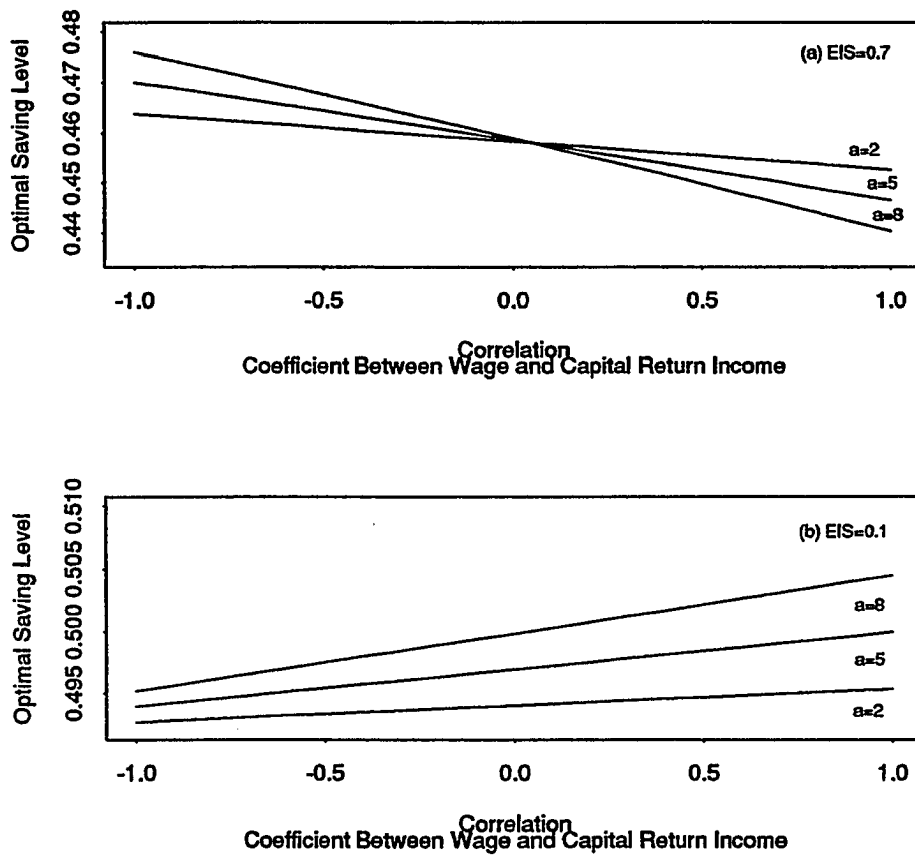
Figure 3.11. Optimal saving path with high wages and $e_2 = 0$.

income correlation rises.

It is interesting to observe that the correlation term can apparently counteract the effect on precautionary saving that a change in the variance of capital returns may induce. For example, a univariate case may exist where an increase in σ_r^2 results in a rise in precautionary saving where the income effect mildly dominates the substitution effect. However, when cast in a multivariate setting with wages and capital income returns highly correlated, we see that precautionary saving may actually fall if expected returns to saving are small compared to expected income in period 2.

Stochastic Endowment and Capital Income

When both second-period endowment income and capital returns are random, it can be shown that the reduced-form first order condition from the basic model is

$$\left(\frac{v_1}{v^e(\tilde{v}_2)}\right)^{-(\delta_1+1)} = \beta \left(\frac{w_2}{w_1}\right)^{b-1} \left((1+\mu_r)-aAw_2^{b-1}(s\sigma_r^2+\sigma_{er})\right)$$

which is identical to the first order condition in the univariate case of stochastic capital returns except for the addition of the new covariance term in the expression for the risk-adjusted gross rate of saving return. Note that the risk-adjusted gross interest rate falls when the covariance between stochastic endowment and capital income in period 2 rises, where again a fall in the risk-adjusted return will make the individual less willing to substitute greater utility in period two for lower utility in period one. This is not surprising since as we have seen before a risk averse individual will interpret a rise in the covariance term as a fall in the certainty equivalent level of second period income relative to the variance in second period income. Precautionary saving is increased to compensate for this drop in expected income.

Implicitly differentiating the first order condition, and noting that $\sigma_{er} = \rho_{er}\sigma_e\sigma_r$,

where ρ_{er} is the correlation coefficient between endowment and capital return income,

reveals that

$$\frac{\partial s}{\partial \rho_{er}} = \frac{C_{10}\left(\left(\delta_1 + 1\right)C_{12} - C_{11}\right)}{U_5}$$

where C_{10}, C_{11}, C_{12}, and U_5 are all positive for reasonable specifications of the model

parameters (see appendix A). Assuming risk averse behavior, it is interesting to note that a

change in the correlation between endowment income and capital return realizations

produces both a substitution and income effect on saving behavior very similar to a change

in either the variance of capital returns or the correlation between wage and capital return

income. Similar to the earlier discussion, an increase in income correlation will generate

precautionary dissaving through a substitution effect unless aversion to intertemporal

substitution is high enough to generate precautionary saving through an offsetting income

effect.

Figures 3.12 and 3.13 exhibit a similar pattern of saving behavior as was shown in

Figures 3.10 and 3.11. Again, the direction of the precautionary saving response to income

correlation depends critically on the EIS parameter and not on the risk aversion parameter,

which merely serves a secondary role in determining the strength of the saving response.

Note that throughout all of the simulations in the chapter, changes in the variance

terms typically produced larger saving responses than changes in the correlation terms,

although the latter responses are not trivial. It is apparent that in the cases involving capital

income risk, the effect of a small change in the variance of an income source on

precautionary saving behavior can conceivably be overpowered by an offsetting effect due

to a large change in the correlation of capital income with another income source, leading

to divergent saving predictions in univariate and multivariate income risk environments.

For example, examination of Figures 3.4(b) and 3.12(b) indicates that a small (< 0.01)

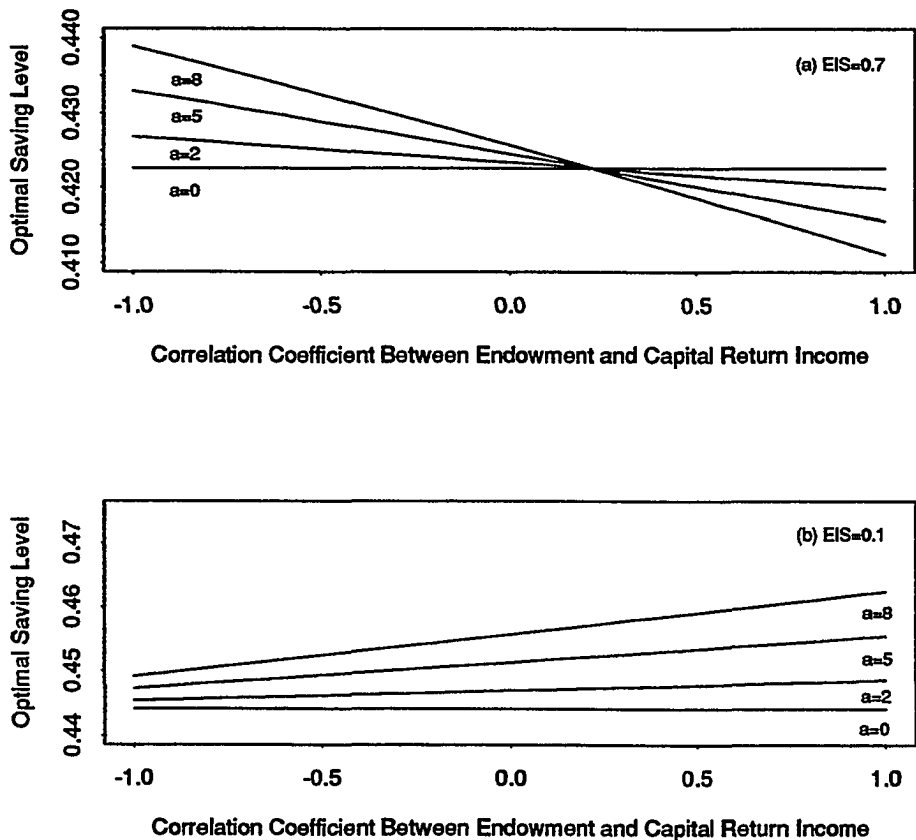increase in the variance of capital returns will reduce saving, but that such a reduction in

Figure 3.12. Optimal saving path with moderate wages and $e_2 = 0.1$.
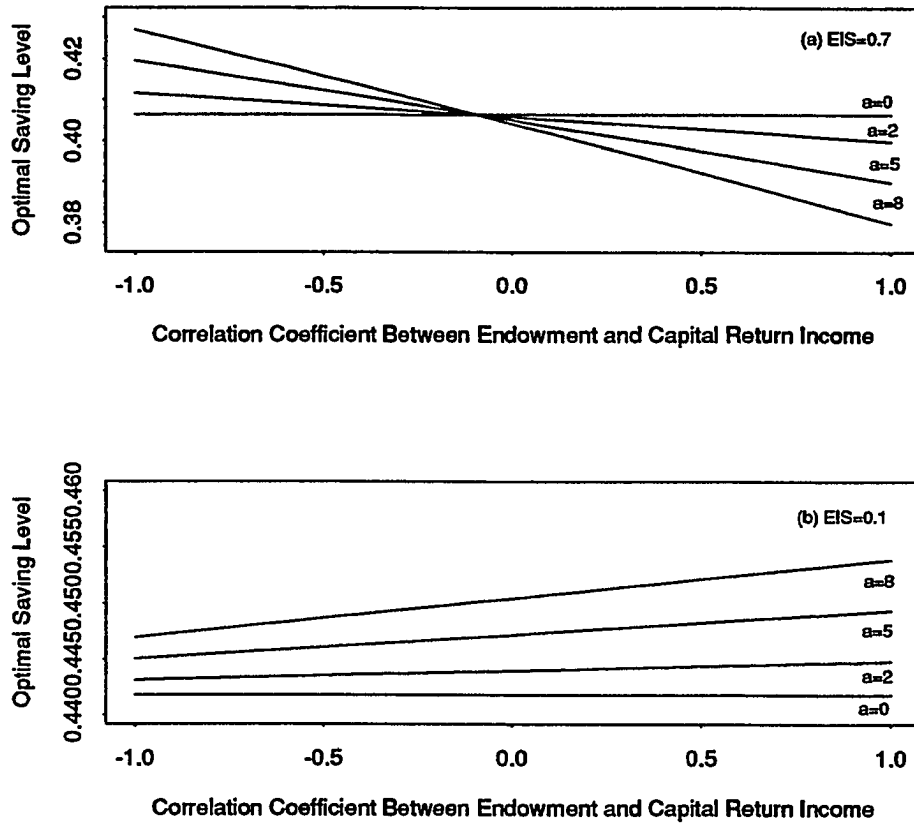
Figure 3.13. Optimal saving path with high wages and $e_2 = 0.1$.

saving would be more than offset if a large (> 0.1) increase in the correlation of capital returns and wages also occurs. As a consequence, a consideration of multiple income risk sources will certainly affect, and may even reverse, the saving implications derived using only univariate income risk models.

As discussed earlier, the consequences of considering multivariate income risk and a correlation effect can dilute the findings using univariate income risk regarding saving behavior under risk, both in terms of the effects of changes in income variances as well as the strong conclusions regarding the fundamental roles of risk aversion and intertemporal substitution aversion in influencing saving and labor supply decisions.

In addition, the importance of the EIS parameter, particularly in the cases involving capital income risk, is clearly shown. Income risk which is endogenous in nature, and therefore affects the intertemporal price of resource transfers, influences factor allocations in a way that depends critically upon attitudes toward intertemporal substitution. However, these attitudes are shown to be irrelevant in the model if income risk is exogenous in nature, and therefore affects only the certainty-equivalent level of wealth rather than the slope of the certainty-equivalent wealth constraint. These findings are model specific and depend on the assumptions of homothetic intertemporal preference and perfect capital markets.

CHAPTER 4: EU AND PRECAUTIONARY SAVING

Introduction

The results in the previous chapter highlight the distinct roles that risk attitude and intertemporal preferences play in influencing the precautionary saving motive under income risk. It is instructive to contrast these results with those obtained with a more traditional intertemporal expected utility (EU) model.

We saw in chapter three that when risk attitude and intertemporal preferences were explicitly separated within the GEU framework used there, either one or the other preference concept prevailed over the other as the fundamental source of the precautionary saving motive, especially in the univariate cases. In particular we saw that the risk attitude parameter was critical to determining the precautionary saving response under endowment income risk, given the assumptions of perfect capital markets and homothetic intertemporal preferences, whereas the EIS parameter primarily determined both the sign and the magnitude of the precautionary saving response under capital return risk as well as the magnitude of the response under wage income risk.

These asymmetric results can not occur by construction within an intertemporal EU model, as we saw in chapter two. The (unfortunate) result has been that precautionary saving behavior and risk attitude are closely linked in the traditional literature[1] at the expense of failing to consider the precise role of intertemporal preferences in motivating saving.

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[1] The literature review in chapter two discusses this issue. Also noted in chapter two was the result that the role of risk aversion in motivating observed intertemporal behavior can never be clearly analyzed using expected utility because of the way preferences are meshed in the expected utility functional.

Here we will examine a fairly representative model of saving behavior in the EU tradition that closely parallels the model in chapter three and also allows for the derivation of analytical solutions that can be compared to the previous results.

The emphasis here will be on the differences between the comparative static results produced in this EU model and the results produced using the GEU model of chapter three, and on how those differences lead to contrasting interpretations of saving behavior. Relatively little attention will be paid to the similarities between the two sets of results.

Basic Model Features

The model used in this chapter is identical to the one used in chapter three with one important difference: the second period risk preference function will now assume the isoelastic power utility form[2]

$$V(\tilde{v}_2(w_2,y_2)) = -\frac{(\tilde{v}_2(w_2,y_2))^{-\delta_1}}{\delta_1}$$

where the function exhibits constant relative risk aversion with a coefficient of relative risk aversion (CRRA) equal to $(1 + \delta_1)$.

As we saw in chapter two, the use of an intertemporal EU framework places strict restrictions on the allowable preferences available to the modeler. If we retain a CES form for the intertemporal utility index (and we must if we wish to derive closed-form saving rules), then the set of risk preference specifications that can be combined with the intertemporal index to construct a two-period cardinal VNM index which is a positive monotone transform of the intertemporal index is quite small. If we further restrict attention to cases where optimization of the expected value of the VNM index yields

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[2] As in the chapter three text it will become convenient later on to drop the arguments of the indirect utility function for ease of notation.

closed-form decision rules then the power function above, or monotone transforms of it, are the only candidates to represent conditional second-period risk attitude.[3]

To keep the comparisons with the previous chapter's results as straightforward as possible, the same intertemporal utility index is used in this chapter. However, to conform with the requirements of the "reduction of compound lotteries" axiom and produce a VNM index that is "linear in the probabilities", the use of certainty equivalents has to be abandoned, and in its place the expected value of the entire VNM index is taken to deal with period 2 income uncertainty.

Thus, in terms of the preference assumptions, the only difference between this model and the GEU model of chapter three is the assumption of constant relative risk aversion as opposed to constant absolute risk aversion. Furthermore, this distinction is not relevant for the issues addressed in this chapter.

In terms of the traditional EU literature on saving behavior, the use of isoelastic risk and intertemporal preferences is fairly standard.[4] Deviations from these preference assumptions are not as common and typically do not produce analytical solutions that can be compared with the results in chapter three.

The certainty case is omitted here since the results would be identical to those in chapter three.

Stochastic Endowment Income

When second-period endowment income is random, it is shown in appendix B that the reduced form first order condition is

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[3] In particular, the negative exponential function used in the GEU model is no longer permissible if we wish to retain a CES intertemporal index - there is no way to combine a CES index and a conditional period 2 exponential function and obtain a VNM index that still retains the intertemporal preferences originally specified.

[4] Van der Ploeg (1992) discusses this in the context of Ricardian equivalence; see also Selden (1979) and Weil (1990).

$$\left(\frac{v_1}{\bar{v}_2}\right)^{-(\delta_1+1)} = \beta\ (1+r)\left(\frac{w_2}{w_1}\right)^{b-1}\left(1+K_e\sigma_e^2\right)$$

where K_e is a function of the model parameters and will be positive for reasonable specifications of them. Thus, the MRS between first-period indirect utility and second-period indirect utility evaluated at the risk neutral expectation of second-period income is a function of the subjective discount factor β, the real relative factor returns, and the variance of second-period endowment income.

This first order condition is slightly different than the corresponding condition derived in chapter three using the GEU model, principally because the MRS term here is not constructed with a certainty-equivalent level of second-period indirect utility, which would implicitly incorporate the variance term for endowment income, but rather includes the risk-neutral expectation of second-period indirect utility. This difference accounts for the explicit appearance of K_e and σ_e^2 in the intertemporal price expression.[5]

Implicitly differentiating the first order condition (see appendix B) shows that

$$\frac{\partial s}{\partial \sigma_e^2} = \frac{(\delta_1+1)D_1}{U_6} > 0 \qquad \forall\ \delta_1 > -1$$

where both D_1 and U_6 will be positive for reasonable parameter specifications.

As in chapter three, a pure income effect drives the level of precautionary saving in response to endowment income risk. Precautionary saving rises (falls) as the variance of second-period endowment income rises (falls), assuming that first- and second-period indirect utility are imperfect substitutes. There will be no precautionary saving if the individual is risk neutral or, equivalently, first- and second-period indirect utility are perfect substitutes,[6] while the precautionary saving motive is strengthened (and the motive

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[5] This distinction will account for the relatively more elaborate intertemporal price expressions in all six cases examined in this chapter.

[6] The implication of risk neutrality necessarily implying the absence of any motive to smooth income over time is a serious criticism of using isoelastic EU modeling. Worse still, it is clear that risk-loving behavior necessarily implies concave intertemporal indifference surfaces!

to use ex post labor supply is weakened) by either an increase in risk aversion or period 1 relative wages, or by an increase in leisure preferences.

However, a significant difference to the results in chapter three appears. Since the δ_1 parameter simultaneously determines the curvature of the second-period utility function as well as the intertemporal utility index, the individual's EIS appears to be relevant here in determining the quantitative response of saving to endowment income risk. In particular, a rise (fall) in the EIS seems to dampen (enhance) the saving response of the individual to endowment income risk according to the comparative static result.

This observation is an illusion and stems from the way risk attitudes and intertemporal preferences are entangled in the construction of the model. Given the assumption of perfect capital markets, a change in the variance of second-period endowment income will, for a risk averse individual, result in a parallel, inward shift of the expected intertemporal wealth constraint.[7] Since the expected wealth expansion path is linear due to the assumption of homothetic intertemporal preferences, how is it possible that the precautionary saving response to a change in endowment income risk should be a function of the curvature of the intertemporal utility index?

Of course, the answer is that it is not possible, but the comparative statics may lead us to believe otherwise. The wealth expansion path will not change simply altering the concavity of the intertemporal indifference surface.[8] Precautionary saving will certainly be a function of how far the wealth constraint shifts, which in turn will be a function of risk attitude or, equivalently, the concavity of the second-period utility function. However, in

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[7] The assumption of perfect capital markets is necessary to regard intertemporal preferences as irrelevant under endowment income risk here and in chapter three - if capital markets were imperfect and risk aversion was present a change in the variance of any future stochastic income source would likely not generate parallel shifts in the wealth constraint, so that the specific curvature of the intertemporal utility index may then influence the precautionary saving response. It would be interesting to compare the results here with results obtained by assuming various forms of imperfections in the capital market.

[8] The slope of the wealth expansion path will be a function of the subjective discount factor β, not the EIS parameter δ_1. Graphically, the discount factor tilts the intertemporal indifference surface, and thus for any given wealth constraint produces a new tangency point, whereas the EIS parameter changes the curvature of the indifference surface without changing the optimal allocation along the existing constraint.

any isoelastic EU model the curvature of the intertemporal and period indifference surfaces are indistinguishable so that risk effects on behavior that are actually attributable to the degree of concavity of only one surface appear to be attributable to both.

Thus, the discussion in chapter three on the roles of risk attitude and intertemporal preferences in determining saving behavior under endowment income risk follows through in its entirety in this chapter as well, despite mathematical results to the contrary. It is risk attitude, and not intertemporal preference, that matters in driving the precautionary saving motive under the assumption of perfect capital markets.[9]

Stochastic Capital Income

When the gross real interest rate is random it is shown in appendix B that the reduced form first order condition is

$$\left(\frac{v_1}{v_2}\right)^{-(\delta_1+1)} = \beta \left(\frac{w_2}{w_1}\right)^{b-1} \left((1+\mu_r) + K_r \sigma_r^2\right)$$

where K_r is a complex function of the model parameters with a sign that is ambiguous for reasonable specifications of its arguments.

Thus the MRS between first-period indirect utility and expected second-period indirect utility evaluated at the risk neutral expectation of second-period income is a function of the subjective discount rate, the relative wage ratio, and the risk-adjusted gross rate of return on the level of first-period saving. Note that the risk-adjusted interest rate falls if aversion to intertemporal substitution rises, and that the form of the first order condition is identical to the corresponding condition in chapter three.

Implicitly differentiating the first order condition (see appendix B) shows that

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[9] This result is also a direct product of using a CES form to represent intertemporal preferences, since a linear wealth expansion path occurs. The use of alternative non-homothetic forms will introduce the EIS parameter as a legitimate source of precautionary saving activity under endowment income risk.

$$\frac{\partial s}{\partial \sigma_r^2} = \frac{(\delta_1 + 1)D_2}{U_7}$$

where U_7 will be positive by the second order sufficient condition and

$$D_2 \equiv \beta \, s \, \mu_y^{-1} \left(\frac{w_2}{w_1}\right)^{b-1} \left[\frac{1}{2}(\delta_1 + 2) \, s(1 + \mu_r) \, \mu_y^{-1} - 1\right]$$

This comparative static result is quite striking in comparison to the corresponding condition using the GEU model. Two things are clearly driving the direction and magnitude of precautionary saving behavior here: (i) the value assumed by the δ_1 parameter (as in the GEU model) and (ii) the magnitude of the risk-neutral expectation of capital income $(s(1+\mu_r))$ relative to the risk-neutral expectation of total income (μ_y) in period 2. Obviously if $\delta_1 = -1$, which indicates both risk neutrality and perfect substitutability between first- and second-period indirect utility, precautionary saving behavior will be absent.

Of greater interest is the observation that if capital income is the only source of period 2 income, so that $s(1+\mu_r) = \mu_y$, then[10]

$$D_2 \equiv \frac{1}{2}\delta_1 \, \beta \, s \, \mu_y^{-1} \left(\frac{w_2}{w_1}\right)^{b-1}$$

We note that if risk preferences are assumed logarithmic (i.e. $\delta_1 = 0$), precautionary saving will not occur since then $D_2 = 0$. This finding mirrors the classic EU result of Samuelson (1969) and Sandmo (1970) who suggest that Bernoulli risk preferences represent a kind of threshold case between which saving is enhanced $(\delta_1 > 0)$ or reduced $(\delta_1 < 0)$ relative to the certainty case. In fact, this result is a special case of the model here since we also note

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[10] This restriction does not necessarily imply that wages must be equal to zero - they may simply be at too low a level to induce a labor supply offer. The absence of a labor market would not change the subsequent analysis, in that the first-order condition and comparative statics would retain their current form except that the relative wage term would disappear.

that if alternative income sources to capital returns are available in period 2, so that $s(1+\mu_r)$ $< \mu_y$, then the threshold will increase.[11]

However, the main difference with the corresponding GEU result in chapter three is the finding here that both risk attitude and intertemporal preferences simultaneously determine the sign of the comparative static expression and the strength of the income effect on precautionary saving of a change in capital return risk. The GEU framework allows for the more plausible interpretation that it is the EIS parameter that primarily determines the sign of the comparative static expression and the strength of the income effect,[12] whereas risk aversion serves a subordinate role in determining the strength of the offsetting substitution effect on precautionary saving behavior.

The GEU interpretation follows since a change in the variance of capital returns will result in a nonparallel rotation of the expected intertemporal wealth constraint for a risk averse individual (unlike the case of endowment income risk) so that the curvature of the intertemporal utility index, as well as the magnitude of the rotation itself, will determine the precautionary saving response.[13]

Thus the dual role of the δ_1 parameter in an isoelastic intertemporal EU model obscures the true nature of the income and substitution effects at work on saving behavior

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[11] As capital income is steadily reduced in importance for expected total second-period income, the threshold level of relative risk aversion will steadily rise above the log specification (i.e. for $s(1+\mu_r)$ = (i) $1/2\ \mu_y$, (ii) $1/4\ \mu_y$, or (iii) $1/10\ \mu_y$, the threshold level of relative risk aversion will become (i) 3, (ii) 7, and (iii) 19). Thus, as capital income is reduced in importance, it becomes more likely for the substitution effect to overpower the income effect on precautionary saving behavior as capital risk rises. No such simple relationship between relative risk aversion and saving behavior occurs using the GEU model of chapter three.

[12] Particularly at low levels of tolerance for intertemporal substitution (i.e. high values of δ_1).

[13] Graphically, risk aversion determines the magnitude of the rotation downward in the expected wealth constraint when capital return risk rises and thus the strength of the negative substitution effect on saving. The degree of aversion to intertemporal substitution determines whether the new optimal utility allocation on the lower expected wealth constraint lies to the left or to the right of the old allocation point and thus the strength of the positive income effect on saving. If the individual is assumed a priori to be risk averse, so that the substitution effect is negative, then precautionary saving under capital risk will be primarily a function of the EIS parameter δ_1, as the comparative static result reveals in chapter three.

when capital return risk is present and it then becomes possible to mistakenly talk about a "threshold" level of risk aversion while ignoring intertemporal preferences.

This confusion is not possible in the GEU model of chapter three which clearly distinguishes between the two critical preference concepts underlying the precautionary saving motive.

Stochastic Wage Income

When second-period wage income is random it is shown in appendix B that the reduced form first order condition is

$$\left(\frac{v_1}{\bar{v}_2}\right)^{-(\delta_1+1)} = \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} (1+r)(1+K_w\sigma_w^2)$$

where $\bar{v}_2$ is the expected level of period 2 indirect utility evaluated at μ_y and K_w is a complex function of the model parameters which will be positive for reasonable values of its arguments. The analysis of this first order condition is analogous to that of the corresponding GEU first order condition, except to note that K_w here is more complex and includes both risk attitude and intertemporal preferences as arguments.

Implicitly differentiating the first order condition we see that

$$\frac{\partial s}{\partial \sigma_w^2} = \frac{D_3 + (\delta_1+1)D_4}{U_8} > 0 \qquad \forall \ \delta_1 > -1$$

where D_3, D_4, and U_8 will all be positive for reasonable parameter specifications. We note the similar form of this comparative static expression to the one derived using the GEU model in chapter three, and also the similar way that a change in the variance of second-period wage income, like a change in the variance of capital returns (and unlike a change in the variance of endowment income in period 2), will rotate the expected intertemporal wealth constraint for a risk averse individual rather than cause it to shift in a

parallel manner. A rotation of the constraint produces an enhanced role for intertemporal preferences in motivating precautionary saving, and it is apparent from a comparison with the corresponding results in chapter three that the EIS parameter δ_1 plays a more significant role, and risk aversion matters less, there in determining the magnitude of the comparative static expression above than here, in an isoelastic EU framework, where risk attitude and intertemporal preferences are de facto equivalent in their importance.[14]

Stochastic Endowment and Wage Income[15]

When both second-period endowment and wage income are jointly random it is shown in appendix B that the reduced form first order condition is

$$\left(\frac{v_1}{\overline{v_2}}\right)^{-(\delta_1+1)} = \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} (1+r)(1+K_e \sigma_e^2 + K_w \sigma_w^2 + K_{ew} \sigma_{ew})$$

where K_e, K_w, and K_{ew} are complex functions of the model parameters and all of which are positive for reasonable values of their arguments.

Implicitly differentiating the first order condition with respect to the correlation coefficient between endowment and wage income (ρ_{ew}) yields

$$\frac{\partial s}{\partial \rho_{ew}} = \frac{(\delta_1+1)D_5}{U_9}$$

where D_5 and U_9 are positive for reasonable specifications of the model parameters. These results contrast with the corresponding findings in chapter three where risk attitude played

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[14] In the corresponding GEU results of chapter three the risk attitude parameter appears within expressions that are very small in magnitude for reasonable parameter specifications, whereas the parameter indicating intertemporal aversion stands apart as a product term for expressions of greater magnitude. This asymmetry disappears in the EU framework.

[15] The three multivariate cases that proceed here basically reflect the same differences with the GEU results in chapter three that were discussed in the preceding univariate cases, so that detailed elaboration will be omitted - where the differences with the univariate cases do occur will be pointed out.

a primary role, and the EIS played a subordinate role, in determining the magnitude of the precautionary saving response to changes in either the variance of endowment income and the correlation between endowment and wage income. Also, the results in chapter three suggested that the EIS played the primary role in determining the magnitude of the precautionary saving response to a change in the variance of wage income in period 2. Here both risk attitude and intertemporal preferences are equally important in determining precautionary saving.

Thus we see that the observations in the three univariate cases carry over into a multivariate setting to demonstrate that the essentially asymmetric roles of risk attitude and time preference in determining the precautionary saving motive are hidden from us in this intertemporal EU framework.

Stochastic Wage and Capital Income

When both wage and capital income uncertainty exists it is shown in appendix B that the reduced form first order condition is

$$\left(\frac{v_1}{\bar{v}_2}\right)^{-(\delta_1+1)} = \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} \left((1+\mu_r) + K_w \sigma_w^2 + K_r \sigma_r^2 + K_{wr} \sigma_{wr}\right)$$

where K_w, K_r, and K_{wr} are complex functions of the model parameters. As noted in the two corresponding univariate cases, K_w will be positive while the sign of K_r is ambiguous for reasonable specifications of the model parameters. In addition, the sign of K_{wr} will also be ambiguous and depend on the same factors as K_r.

Implicitly differentiating the first order condition with respect to the correlation coefficient between wage and capital return income (ρ_{wr}) yields

$$\frac{\partial s}{\partial \rho_{wr}} = \frac{D_6 + (\delta_1+1)D_7}{U_{10}}$$

where U_{10} is positive while the signs of D_6 and D_7 are both ambiguous depending on the magnitude of the risk-neutral expectation of second period capital income relative to the risk-neutral expectation of second period total income. In the special case of Bernoulli risk preferences, we obtain the particularly simple result that both coefficients are positive (negative) if expected capital income is more (less) than one half expected total income in period 2, so that precautionary saving rises (falls) as the correlation between the two income sources strengthens.

In the GEU model results we saw that the precautionary saving behavior of a risk averse individual facing changes in the variance of either wage or capital return income, or a change in the correlation of wage and capital returns, depended critically upon the EIS parameter and to a lesser extent upon risk attitude. In particular, the EIS played a dominant role in determining the magnitude of the saving response to wage income risk in this multivariate setting, and primarily determined both the sign and the magnitude of the saving response to capital return risk as well as the correlation of factor returns.[16] This distinction disappears within the EU result here.

Stochastic Endowment and Capital Income

When both second-period endowment income and capital returns are random it can be shown that the reduced form first order condition from the EU model is

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[16] The relative size of expected capital income also played a critical role in determining the multivariate comparative static signs in the GEU model. In particular, if $s(1+\mu_r) = \mu_y$ then a precautionary saving motive, given a change in capital return risk, existed for any value of δ_1 greater than -1, but for $s(1+\mu_r) < \mu_y$ a precautionary saving motive existed only at higher values of δ_1. Thus, Werner's (1990, p. 47) observation that the presence of joint wage and capital income risk should not qualitatively affect her results is not entirely accurate. Similar findings occur with respect to the role of the correlation of factor returns on precautionary saving, although saving is more sensitive to the relative size of expected capital income when the correlation term changes than if the variance of capital return changes.

$$\left(\frac{v_1}{v_2}\right)^{-(\delta_1+1)} = \beta \left(\frac{w_2}{w_1}\right)^{b-1} \left((1+\mu_r)+K_e\sigma_e^2+K_r\sigma_r^2+K_{er}\sigma_{er}\right)$$

where K_e, K_r, and K_{er} are complex functions of the model parameters. As noted earlier, K_e will be positive for reasonable specifications of its arguments while K_r will be either positive or negative depending upon both δ_1 and the relative importance of capital income in period 2. Similarly, K_{er} will also be positive or negative depending upon the magnitude of δ_1 and capital income.

Implicitly differentiating the first order condition with respect to the correlation coefficient between endowment and capital return income (ρ_{er}) yields

$$\frac{\partial s}{\partial \rho_{er}} = \frac{(\delta_1+1)D_8}{U_{11}}$$

where the sign of U_{11} is positive while D_8 is ambiguous. Assuming $\delta_1 > -1$ (i.e. ruling out risk neutrality and perfect substitutability), the expression indicates that the direction of precautionary saving in response to a change in the correlation between endowment income and capital returns depends critically upon the magnitude of capital income. As with the case of joint wage and capital income, an increase in the correlation of the two income sources enhances (dampens) precautionary saving in the Bernoulli case only if the risk-neutral expectation of capital income is greater (less) than one half the risk-neutral expectation of second-period income - no such simple results obtain when risk attitude and intertemporal preference are separately parameterized, however, as chapter three demonstrates.

In the corresponding multivariate GEU model we saw that the sign of the comparative static expression depended primarily upon the individual's EIS parameter, which does not hold here. Again, an asymmetric treatment of risk attitude and intertemporal preferences is lost using an intertemporal EU framework, whether in a univariate or multivariate setting.

CHAPTER 5: GEU AND ASSET PRICING

Introduction

This chapter examines the usefulness of a class of GEU models proposed by Epstein and Zin (1991), which represent an infinite-horizon extension of the two-period model examined in chapter 3, in intertemporal asset pricing applications. Using actual U.S. consumption and market return data, the models will be tested using the diagnostics proposed by Hansen and Jagannathan (1991).

The chapter is organized as follows. This opening section will discuss intertemporal asset pricing modeling and the associated Euler equations as first proposed by Lucas (1978) and Breeden (1979) and also the recursive utility methodology of Epstein and Zin, which can be incorporated into a Lucas-type framework. In addition, an overview of the Hansen and Jagannathan (HJ) "volatility" bounds test of asset pricing models will follow, as will a compact literature review. The second section will describe the three related GEU models of Epstein and Zin, Kandel and Stambaugh (1991), and Kocherlakota (1995) that will be tested, while section three will discuss in detail the consumption and asset return data used in the analysis. Section four will present the estimation procedure and results, with the conclusions discussed in section five.

Euler Equations and Intertemporal Asset Pricing Models

Lucas (1978) uses a stochastic pure endowment economy with one good and identical consumers to describe relationships between aggregate consumption and asset returns which must hold in a competitive general equilibrium. Discussing the questions of

both the existence and uniqueness of equilibrium asset prices in such an economy, he characterizes a unique equilibrium asset price function with his equation (6):

$$u'(\sum_i y_i) p_i(y) = \beta \int u'(\sum_i y_i')(y_i' + p_i(y')) dF(y', y) \qquad i = 1, \dots, n$$

where u is current utility, y_i is the output level of unit "i", a consumption/investment good, p_i is the price of unit "i", β is a discount factor on future consumption, and F(y',y) is the transition function defining the motion of output y over time. This is a stochastic Euler equation which loosely equates the marginal rate of substitution of current for future consumption to the expected market rate of return on any security "i", which effectively transforms current consumption into future consumption. The set of all n assets represents an agent's opportunity set for transferring consumption between periods t and t+1 and beyond.

This construction, together with work by Breeden (1979), yielded the consumption-based equilibrium model of asset pricing which is today predominant in macroeconomics and important in finance:

(5.1) $q_t = E_t(m_t \, y_{t+1})$

where q_t denotes the asset price in period t, E_t is a conditional expectations operator, m_t denotes the intertemporal marginal rate of substitution (IMRS) between current and future consumption, and y_{t+1} denotes the asset payoff in period t+1. The IMRS function is characterized as:

(5.2) $m_t = \beta \left(\dfrac{MU(\tilde{c}_{t+1})}{MU(c_t)} \right)$

where β denotes the subjective time discount factor, MU( ) denotes marginal utility, and c_t denotes consumption in period t. The tilde over period t+1 consumption in the IMRS expression signifies that this consumption is stochastic from the vantage point of period t decision-making. Essentially equation (5.1) states that the price of an asset in period t can be calculated as the discounted value of its expected payoff one period later, where the "discounting" factor is the IMRS. The IMRS itself is calculated as the product of an

investor's subjective discount rate and her subjective marginal rate of substitution between a known level of current consumption and a generally unknown level of future consumption.

An extensive amount of theoretical and empirical research has been published during the last fifteen years investigating the properties and usefulness of equation (5.1) in understanding asset price movements in actual markets. Numerous elaborations on the basic model are possible, and they are briefly discussed in the conclusion of the chapter. A primary focus of current research in this area has been to develop stochastic laws of motion for the model variables and functional forms of the representative investor's utility index which allow the model to generate enough aggregate risk to explain the volatility of actual asset returns observed in real world markets. The relatively smooth time profile of consumption data in the U.S. compared to the time profile of most asset returns has been a source of great concern for advocates of consumption-based asset price modeling.

The empirical force of this model lies in the (overidentifying) restrictions it places on the time paths ("law of motion") of consumption and asset returns. The restrictions dictate that the series must covary in such a way that the product of the proposed IMRS and each asset return has a conditional mean equal to unity, which represent conditional moment restrictions that constrain the conditional means of nonlinear functions of the data. Method of moments and generalized method of moments (GMM) parametric tests of the restrictions implicit in the model focus on whether (i) the law of motion or (ii) the functional form of the IMRS are misspecified. The overidentifying restrictions are usually rejected when tested using data on consumption growth and asset returns, particularly when a strictly Gaussian law of motion is assumed and time-additive, isoelastic EU is attributed to a representative agent.

Because the Lucas and Breeden models use an intertemporal EU framework in their analysis they cannot disentangle risk attitudes from intertemporal substitution preferences. It is a well-known constraint intrinsic to time-additive, isoelastic expected utility that the elasticity of intertemporal substitution (EIS), which determines the curvature

of the between-period intertemporal utility index, is the inverse of the coefficient of relative risk aversion (CRRA), which determines the curvature of the within-period atemporal utility index. Thus, neither researchers can provide a clear interpretation of their comparative static results (see Lucas, p. 1441) or a clear understanding of the determinants of asset prices in their models. Using the class of models examined here allows a clean separation of these two distinct preference concepts and thus allows a better understanding of asset price movements.

In this chapter we will be interested in the properties of the IMRS expression in equation (5.2) when it is derived within a GEU recursive framework - in particular we are interested in whether moment restrictions on equation (5.2) implied by historical U.S. asset return data can be reconciled with moment restrictions implied by historical U.S. consumption and market return data using a GEU index. We will use the nonparametric testing procedure of HJ to determine whether this specification of a representative agent's intertemporal utility index can significantly improve upon the performance of earlier EU models in explaining asset market behavior.

Recursive Utility

It is clear that, with the same preferences over present and future consumption, two households with identical income will save the same in a certain environment, but that in a world of uncertainty this need not be true. This distinction between intertemporal substitution preference and risk attitude cannot be achieved cleanly within the time-additive von Neumann-Morgenstern isoelastic EU framework by a concave transform of the objective function for two reasons: (i) the presence of uncertainty within the objective function after the concave transform dictates that both risk attitude and substitution preferences will be captured within the curvature parameter used in the transformation - to have the parameter free of any influence from risk attitude all uncertainty must first be

removed from the objective function, and (ii) as Epstein and Zin (1989, pp. 951-952) point out with regard to the results of Kihlstrom and Mirman (1974), such an approach has an unappealing feature in an intertemporal framework: preference orderings will generally depend on past consumption values with the possible exception of an exponential transform. This implies that, in general, attitudes toward future gambles are changing with the passage of time and that present consumption plans are time-inconsistent.

As a result, numerous researchers (Epstein, 1988, Epstein and Zin, 1989,1991, Weil, 1990,1993, Bufman and Leiderman, 1990, Kandel and Stambaugh, 1991, Prasad, 1991, Obstfeld, 1994a, 1994b, Campbell, 1993, Hung, 1994, and Epstein and Melino, 1995, to name a few) have moved away from the intertemporal EU framework and instead use versions of the recursive utility framework developed by Kreps and Porteus (1978,1979) in order to achieve the desired separation between intertemporal substitution and risk aversion. A great deal of attention has been focused on interpreting stochastic interest rates, asset prices, and the capital asset pricing model (CAPM) with generalized recursive isoelastic preferences (Epstein and Zin, 1989,1991, Giovannini and Weil, 1989, Attanasio and Weber, 1989, and Weil, 1989). Epstein (1988) investigates equilibrium asset prices in a Lucas-type exchange economy where preferences are represented by a Kreps-Porteus (KP) functional rather than an EU functional to show how a clean separation of time and risk preferences allows a better understanding of the determinants of asset prices.

A recursive utility framework also deviates from the customary assumption of indifference to the temporal resolution of uncertainty (i.e. the axiom relating to the reduction of compound lotteries - see chapter one for further discussion).

At this point a reasonable question to ask is: why test a recursive utility framework instead of the two-period framework adopted in chapter three? The theoretical answer, as again noted by Epstein and Zin (1989), is that use of these preferences (which are derived from Selden's (1978) ordinal certainty equivalence (OCE) preferences) in an intertemporal asset pricing model such as the CAPM introduces intertemporal consistency

problems and generates Euler equations applicable only to a naive consumer/investor who continually ignores the fact that plans formulated at any time will generally not be carried out in the future. The need to derive stationary, intertemporally consistent optimal time-profiles of consumption requires the use of utility functions based on a recursive structure. Since the two-period utility index used in chapter three is not based on a recursive structure, tests of its ability to explain actual asset market behavior would imply that market participants must be irrational.

The recursive model as envisioned by Epstein and Zin (1991), which is a straightforward extension of work by Selden and of the Kreps and Porteus formulation of the space of temporal lotteries to an infinite-horizon framework, can be outlined as follows. Consider an infinitely-lived representative agent who derives utility from a single consumption good c. At time t current real consumption c_t is known but in general future real consumption is unknown. The agent's utility is recursive in the sense that utility today satisfies the recursive relation

(5.3) $U_t = W(c_t, \mu(U_{t+1} \mid I_t))$ for all $t \geq 0$

where U_t denotes current utility, W is an aggregator function over current real consumption and a risk-adjusted index of the future, and $\mu(U_{t+1} \mid I_t)$ denotes the conditional certainty equivalent of random future utility. This utility form generalizes the recursive structure of Koopmans (1960) in a deterministic setting and Kreps and Porteus in a stochastic setting. The aggregator function W can assume a variety of forms - one of the more common is a constant elasticity of substitution (CES) specification such as

$W(c,\mu) = [(1-\beta)c^\rho + \beta U^\rho]^\rho$

where $0<\beta<1$ is a subjective discount factor and $0\neq\rho<1$ is a substitution parameter. The certainty equivalent μ assigns a nonnegative real number to the random real consumption variable(s) in its domain. It satisfies the requirement that $\mu(b) = b$ for any certain nonnegative real b and $\mu(\lambda z) = \lambda\mu(z)$ for all $\lambda>0$ and for all random variables z in the domain of μ. This second requirement essentially imposes constant relative risk aversion

(for analytical tractability). A common functional form in empirical asset pricing models is the EU formulation

$$\mu(z) = (Ez^\rho)^{1/\rho}$$

which implies in a CES utility function that

$$U_t = [(1-\beta)E_t \sum_{i=0}^{\infty} \beta^{\,i} c_{t+i}^\rho]^{1/\rho}$$

The interesting and important consequence of such a model for asset pricing purposes, as already discussed, is that since μ depends on the substitution parameter ρ, both risk aversion and intertemporal substitution are mixed in a single parameter. Theoretical speculation has led many researchers to suggest that this (mis)specification of utility contributes to the poor empirical performance of intertemporal expected utility models in explaining aggregate real consumption and asset real return data.

A more natural generalization for the certainty equivalent functional introduces a separate parameter to model risk aversion

$$\mu(z) = (Ez^\alpha)^{1/\alpha}$$

where $0 \neq \alpha \leq 1$. When α is not constrained to be equal to ρ, a change in the specification of μ only affects risk attitudes. Of course, alternative specifications, including nonparametric ones (see Kandel and Stambaugh, 1991), can be contemplated for the modeling risk attitudes.

Finally, regarding the functional form of recursive utility itself, in the case of a certain real consumption sequence $(c_0, c_1, \ldots c_t, \ldots)$, U_t can adopt the CES form shown above

$$U_t = [(1-\beta) \sum_{i=0}^{\infty} \beta^{\,i} c_{i+t}^\rho]^{1/\rho}$$

where $(1-\rho)^{-1}$ is then the EIS parameter. For any fixed deterministic sequence $(c_0, c_1, \ldots c_t, \ldots)$ and for any (scalar) random rescaling s of future real consumption resolved at time 1, the utility of the random real consumption sequence $(c_0, sc_1, \ldots sc_t, \ldots)$ equals

$$[(1-\beta)c_0^\rho + \beta\mu^\rho(s)U_1^\rho]^{1/\rho}$$

where

$$U_1 = [(1-\beta)\sum_{i=0}^{\infty} \beta^i c_{i+1}^\rho]^{1/\rho}$$

which implies that $(c_0, s'c_1, ...s'c_t, ...) \succ (c_0, sc_1, ...sc_t, ...) \Leftrightarrow \mu(s') > \mu(s)$, which describes the sense in which it is appropriate to interpret μ as representing an agent's risk attitude.

Hansen-Jagannathan (HJ) Volatility Bounds

In Hansen and Jagannathan (1991) a diagnostic tool is constructed for a wide variety of intertemporal asset pricing models which involves the calculation of unconditioned mean-standard deviation ("volatility") frontiers for the IMRS of consumers (i.e. $\{E(m_t), \sigma(m_t)\}$-pairs) and which represents a more general bound than the variance bounds constructed by Shiller (1982,1987) and Hansen (1982). These frontiers give the lower bound on the standard deviations of the IMRS as a function of its mean value which is consistent with historic asset real return data. Moment restrictions on the IMRS implied by theoretical asset pricing models can then be compared to these frontiers, where failure of moment coincidence is an indication of a misspecified model. This nonparametric approach to evaluating asset pricing models complements existing parametric approaches such as GMM that are prevalent in the literature. The moment restrictions of the IMRS implied by asset price and return data are useful as a first-level diagnostic for evaluating virtually any intertemporal asset pricing model in which it is possible to calculate the first two moments of the IMRS implied by the model, as the HJ bounds involve minimal restrictions on a feasible IMRS: (i) $m_t > 0$ for any period t, and (ii) the unconditional expectation of equation (5.1) holds for any period t.

The bounds are widely used with respect to testable implications of "general" dynamic asset pricing models. In evaluating the validity of common tests of the consumption-based CAPM (C-CAPM), Kocherlakota (1990) and Burnside (1994) find that the small sample properties of the HJ test are better than those of other tests, including the

GMM procedure studied by Hansen (1982), in that the frequency of type I errors (i.e. overrejecting a true model) are more likely with GMM due to the burden of actually estimating parameter values rather than evaluating known parameters as in the HJ procedure. In particular, the HJ test is useful if an econometrician wishes to test the hypothesis that a representative agent with specific preference parameters (e.g. $\beta = 0.99$, CRRA = 2, EIS = 0.15) in a specific asset pricing model can reasonably explain asset market behavior in an economy. Thus, the econometrician knows the true preference parameters of the representative agent and does not have to estimate them. Since the data set used in this study is relatively small (involving 89 quarterly asset real return, real consumption growth, and real market return observations), the HJ test seems an appropriate diagnostic for the theoretical models examined here.

Of interest in this study are the HJ bounds constructed from the (time-averaged) quarterly holding period (ex post) real returns on U.S. Treasury bills (see Figure 6 in HJ). The quarterly nominal returns used represent percentage changes of bond prices on 3-, 6-, 9-, and 12-month discount bonds for the period 1964:III through 1986:IV. Real returns were obtained by deflating the nominal returns using a monthly implicit price deflator for consumption of nondurable goods and services; thus, four monthly time series of 3-month holding period returns were constructed using the monthly price data on the four types of bonds and are incorporated into HJ figure 6.

HJ claim that the resulting IMRS volatility bounds implied by return data on U.S. Treasury bills represent a "significant challenge" to several classes of asset pricing models. Subsequent research has confirmed their claim as the implied volatility of the IMRS using asset market data cannot be reproduced using the standard theoretical consumption-based capital asset pricing model (C-CAPM) with expected utility preferences

$$q_t = \beta \left(\frac{\tilde{c}_{t+1}}{c_t} \right)^{-(1-\alpha)} y_{t+1}$$

where (1-α) denotes the CRRA, unless astronomical risk aversion is assumed by a representative investor (see HJ, 1991). As discussed above, the main problem with using consumption-based asset pricing models is that the time profile of consumption data, such as the gross real consumption growth rate in the above equation, is too smooth compared to the volatility of asset return data in the U.S. Increasing the CRRA in the expression can magnify the variance in the consumption growth rate data, but matching asset return volatility requires what some researchers regard as too high a level of risk aversion to be plausible. This problem is related to the so-called "equity premium puzzle" introduced by Mehra and Prescott (1985), who observe that the large average historic equity premiums (i.e. equity returns above the risk-free interest rate) paid in the U.S. over the last one hundred years cannot be rationalized in a Lucas-type equilibrium model, such as the one above, if one restricts the CRRA to be "reasonable" (less than ten). A similar situation arises in the "risk-free rate" puzzle of Mehra and Prescott and Weil (1989) which involves explaining why the historic average interest return on 3-month Treasury bills (which are regarded as essentially risk-free assets in the literature) has been so low, especially if one assumes a representative agent who is only moderately risk averse (CRRA<10) but extremely averse to intertemporal substitution in consumption, as the findings of Hall (1988) suggest. Both puzzles arise essentially because the standard theoretical models do not generate enough aggregate risk under these types of preference restrictions.

The goal in this study is to determine whether the class of GEU models introduced by Epstein and Zin, which allows independent parameterization of intertemporal substitution and risk aversion preferences, will produce IMRS moments using consumption and market return data over the period 1964:III to 1986:IV that will fall within the volatility bounds constructed by HJ using Treasury bill return data. In other words, we wish to determine whether or not independently specified but "reasonable" time and risk preference parameters, together with historical and seasonally unadjusted U.S. aggregate quarterly real consumption and market return data, can be used in a GEU

framework to produce moment restrictions on the IMRS of a representative agent which are consistent with moment restrictions implied by historical U.S. asset return data. If this is possible it suggests that theoretical models which fail to separate risk and time preferences in the utility index (such as in time-additive EU models) are discarding relevant information in their attempt to explain asset price movements and asset market behavior in general.

The general formulation of the HJ bounds on the standard deviation of the IMRS can be outlined as follows (Gallant et al., 1990, derive conditional counterparts to the work of HJ). Assume that a risk-free asset exists where the holder receives a unit payoff in every state. Let m_t denote the (scalar) one-period IMRS and let q_t^f denote the (observed) price of the risk-free asset. Then:

$$q_t^f = E\,[m_{t+1} \mid I_t]$$

$$\text{std}\,[q_t^f] = \text{std}\,[E\,[m_{t+1} \mid I_t\,]] < \text{std}\,[m_{t+1}]$$

where std denotes the standard deviation. The inequality in the second condition follows because $E\,[q_t^f]$ is the conditional expectation of the random variable m_{t+1}. Thus, std $[q_t^f]$ provides a lower bound on std $[m_{t+1}]$, and the variability of the observed risk-free price allows a test of a specific economic model of m.

More generally, let y denote a vector of returns on assets with payoffs one period hence and let q denote the corresponding asset price vector. Standard asset pricing models give rise to the following unconditional pricing relation:

$$E\,[my] = \phi$$

where ϕ denotes a vector of ones of the same dimension as y. This relation is the Euler equation restriction on the covariance of the two series m and y. Assume that $E\,|\,m^2\,| < \infty$, $E\,|\,y^2\,| < \infty$, $E\,|\,yy'\,|$ is nonsingular, and $E\,|\,q\,| < \varepsilon$. Although m cannot be calculated directly from the return data, imagine constructing its population least-squares projection onto y, denoted $m^* = y'\delta$, and require it to satisfy the restriction $E\,[m^*y] = \phi$. Then $E[yy'\delta] = \phi$ and $\delta = E[yy']^{-1}\phi$, which suggests that $m^* = y'\,E\,[yy']^{-1}\phi$. Thus we can construct

a least-square estimate of m using the unconditional second moment of y. Note that $E[m^*] = E[m] = E[q_t^f]$ and std $[m^*] \leq$ std $[m]$ since m^* is based on a projection onto y. (note: HJ also discuss more realistic cases in which there is no risk-free asset and how to exploit the restriction that $m > 0$).

In applications population moments are estimated by sample moments. Let m_T^* denote the estimator of m^* based on T observations. An assessment of a candidate IMRS can be made by comparing its standard deviation to std $[m_T^*]$. A metric can be used in comparisons of different candidate IMRS's.

Literature review

Despite an extensive search, no published studies were found which examine the IMRS moment properties of recursive GEU modeling similar to Epstein and Zin (1991) and test them against constructed HJ volatility bounds for any asset or group of assets.

Numerous parametric tests (especially GMM) of the GEU modeling used here have been published with mixed results. Kocherlakota (1990), Jorion and Giovannini (1993), and Kocherlakota (1995) argue that GEU will not improve upon the poor performance of EU models in explaining consumption and asset return data in the United States, whereas Epstein and Zin (1991), Hung (1994), and Epstein and Melino (1995) find evidence that GEU represents a significant step toward explaining the relationships between asset market data and economic aggregates.

One consequence of the generalization from EU to recursive utility (as we will shortly see) is the reemergence of the market return as a factor in explaining excess mean returns on different assets. Thus, both consumption and the market return enter into the covariance that defines systemic risk for an asset in the recursive GEU model, so that GEU represents a blending of standard results from both the intertemporal C-CAPM and the static CAPM. Epstein and Zin (1991) use GMM analysis to show that their GEU model

performs better than standard EU asset pricing, but note that the GEU results are sensitive to the consumption measure choice and the instrumental variables used to estimate model parameters. Emphasizing the controversy, Jorion and Giovannini concur with the Epstein and Zin results about the sensitivity issue, but also show by using maximum-likelihood estimation as well as GMM that GEU does not improve the fit of asset models to U.S. data compared to the standard EU, nor does the more general GEU specification help to better predict the cross-sectional variations in expected returns.

When using GEU one must be careful as to the assumptions made about the law of motion for the relevant variables. Kocherlakota (1990a) correctly points out that when consumption growth rates are assumed i.i.d. (a common but counterfactual assumption in the literature) in an asset pricing model which adopts GEU preferences, the model will not have any more explanatory power over asset price data than models using EU preferences. This is so because asset prices reveal only first-order restrictions on investor preferences whereas the separation of risk and time preferences in a GEU utility index represents a second-order restriction on preferences in that it involves the curvature of the within-period and between-period indifference surfaces. There is no theoretical reason to expect second-order preference restrictions to be manifest in asset price movements under an i.i.d. assumption.

The GEU model has been used in attempts to explain the equity premium puzzle introduced by Mehra and Prescott (1985) and the risk-free rate puzzle of Weil (1989). The two puzzles arise because of the inability of standard intertemporal asset pricing models to generate enough aggregate risk, when "reasonable" levels of risk aversion are assumed, to explain why historic average equity returns in the U.S. have been so much larger than the average risk-free interest rate, and why the average historic risk-free rate has been so low. Weil finds that GEU is an unsatisfactory approach for both puzzles. He claims that the premium on equity securities depends "almost exclusively" on the CRRA, so that simply relaxing time-additive EU restrictions on tastes cannot resolve the equity premium puzzle. Similarly, he rejects the idea that using GEU can explain the extraordinarily low level of

risk-free interest rates in the U.S. He points out that if the CRRA is constrained to be at a "reasonable" level (Weil suggests a value of one), then the EIS in a GEU model must be far greater than the results of Hall's (1988) study which strongly infer that the EIS in the U.S. should be around 0.1. (however, Hall's result has been criticized; see Weil, 1990). Furthermore, he questions why the risk-free rate in the U.S. should be so low if individuals are as averse to intertemporal substitution as Hall suggests (see section five for a possible explanation).

Kocherlakota (1990b, 1995, pp. 17-18) likewise rejects GEU as a solution to the equity premium puzzle, but for a surprising reason: he suggests that the puzzle arises solely because of economist's prior beliefs about the appropriate level of relative risk aversion for individuals, which tend to range anywhere from two to six in value (Obstfeld, 1992). Kocherlakota finds that values as high as twenty are not unreasonable from either theoretical or empirical grounds (see especially Kocherlakota, 1990b and Kandel and Stambaugh, 1991), so that models incorporating GEU (or time-nonseparable utility, or incomplete markets) are missing the crucial point as he sees it - the key is to move away from preconceptions of the "appropriate" level of risk aversion. When these preconceptions are overcome, the equity premium puzzle disappears. Unlike Weil, however, Kocherlakota does see GEU as the solution to the risk-free rate puzzle, since Kocherlakota is willing to permit CRRA values far greater than one. By allowing both the CRRA and the EIS to be high simultaneously (something not possible using EU preferences), the low risk-free rate in the U.S. can be theoretically explained.

As mentioned earlier, the failure of an intertemporal asset pricing model to fit actual market data can be explained as either a misspecification of the law of motion governing the model variable(s) or a misspecification of the form of the intertemporal utility index. A reasonable approach to the problem is taken by Hung (1994), who assumes that both types of misspecification jointly contribute. He shows that using GEU preferences in a general equilibrium asset pricing model in which stochastic consumption growth and stock dividends follow a bivariate Markov switching process can generate

IMRS values with first and second moments consistent with both the risk-free rate and the risk premium in U.S. data.

Models

In this section we will provide compact presentations the intertemporal asset pricing model of Epstein and Zin (1991) which incorporates recursive GEU preferences in the utility index, as well as two other models appearing in Kandel and Stambaugh (1991) and Kocherlakota (1995) which are derived from the Epstein and Zin class of models and use the same variables to characterize a representative agent's IMRS. Although the three models impose the same moment restrictions on the theoretical IMRS, each set of authors offers different interpretations of the basic model which will be commented upon in this section.

Model of Epstein and Zin (1991)

Consider an infinitely-lived representative agent who receives utility from a single consumption good. Lifetime utility is given by

$$U_t = W(c_t, \mu_t [\tilde{U}_{t+1} \mid I_t])$$

where interpretations of each variable were given earlier following equation (4.3). As indicated earlier, these preferences are both consistent over time and stationary (Epstein and Zin (1991)).

Assume the aggregator function has the following homogeneous form:

$$W(c,z) = [(1-\beta)c^\rho + \beta z^\rho]^{1/\rho} \qquad \text{for } 0 \neq \rho < 1$$

$$W(c,z) = (1-\beta)\log(c) + \beta\log(z) \qquad \text{for } \rho = 0$$

where β denotes a discount factor and $(1-\rho)^{-1}$ denotes the EIS. Furthermore, assume the certainty equivalent functional has the following homogeneous form:

$\mu(x) = (Ex^\alpha)^{1/\alpha}$ for $0 \neq \alpha < 1$

$\log[\mu] = E(\log(x))$ for $\alpha = 0$

where x denotes a random variable, $(1-\alpha)$ denotes the CRRA, and E is an expectations operator. Assuming $\alpha \neq 0$ and $\rho \neq 0$, these functional assumptions lead to the recursive structure for intertemporal utility given by

$U_t = [(1-\beta)c_t^\rho + \beta(E_t U_{t+1}^\alpha)^{\rho/\alpha}]^{1/\rho}$

where E_t is a conditional expectations operator given I_t and U_{t+1} is stochastic period $t+1$ utility. When $\alpha = \rho$ (i.e. when CRRA = 1/EIS) we get

$$U_t = [(1-\beta)E_t \sum_{i=0}^{\infty} \beta^i c_{t+i}^\alpha]^{1/\alpha}$$

which is the familiar EU specification. Given that the agent's wealth evolves according to

$A_{t+1} = (A_t - c_t)\omega_t \tilde{R}_t$

where A denotes the consumption good stock, ω_t denotes an N-vector of portfolio weights, and $\tilde{R}_t$ denotes an N-vector of random asset real returns, we can solve for the set of necessary conditions for the joint consumption and portfolio choice decisions (i.e. the Euler equations) in terms of observable variables. Maximizing the following Bellman equation with respect to consumption (where A_{t+1} and I_{t+1} are stochastic variables)

(5.4) $J(A_t, I_t) = \{(1-\beta)c_t^\rho + \beta[E_t(J(A_{t+1}, I_{t+1}))^\alpha]^{\rho/\alpha}\}^{1/\rho}$

subject to the budget constraint above yields the Euler equation for optimal consumption decisions (when $\alpha \neq 0$ and $\rho \neq 0$):

(5.5) $E_t \left[\beta \left(\dfrac{\tilde{c}_{t+1}}{c_t} \right)^{\rho-1} \tilde{M}_{t+1} \right]^\gamma = 1$

where M_{t+1} denotes the gross return on the optimal ("market") portfolio and $\gamma = \alpha / \rho$. Maximizing equation (5.4) with respect to ω_t yields the Euler equation for optimal portfolio selection:

$$(5.6) \quad E_t\left[\left(\frac{\tilde{c}_{t+1}}{c_t}\right)^{\gamma(\rho-1)} \tilde{M}_{t+1}^{\gamma-1}(\tilde{R}_{j,t+1} - \tilde{R}_{1,t+1})\right] = 0$$

for $j = 2,....N$ and where $R_{i,t+1}$ denotes the gross return on asset "i" in period t. We can combine equations (5.5) and (5.6) to obtain

$$(5.7) \quad E_t\left[\beta^{\gamma}\left(\frac{\tilde{c}_{t+1}}{c_t}\right)^{\gamma(\rho-1)} \tilde{M}_{t+1}^{\gamma-1} \tilde{R}_{t+1}\right] = 1$$

which implies that for this recursive model

$$(5.8) \quad m_t = \left[\beta^{\gamma}\left(\frac{\tilde{c}_{t+1}}{c_t}\right)^{\gamma(\rho-1)} \tilde{M}_{t+1}^{\gamma-1}\right]$$

which is valid regardless of the information set of the representative investor. Note from equation (5.7) the amount of structure which is placed on the law of motion for the gross real consumption growth rate, the real market return, and each individual asset return - all three must covary in such a way that their expected product each period is equal to unity.

Using equation (5.8) and incorporating historical real consumption and market return data, we are interested in what values of $(1-\rho)$ and $(1-\alpha)$ will produce theoretical IMRS moment restrictions over the period 1964:III to 1986:IV that fall within the HJ volatility bounds implied by data on U.S. Treasury bills (see section 1). It is obvious that this GEU specification for the IMRS represented by equation (5.8) cannot perform worse than the standard C-CAPM model which uses an expected utility index since it is included as a special case; i.e. when $\alpha = \rho$, then $\gamma = 1$, $\sigma = (1-\alpha)^{-1}$, then (5.8) reduces to

$$(5.8.1) \quad m_t = \left[\beta\left(\frac{\tilde{c}_{t+1}}{c_t}\right)^{-(1-\alpha)}\right]$$

Thus in the standard C-CAPM the moments of the IMRS are determined by the stochastic properties of the gross growth rate in real consumption as well as the risk attitudes of the agent. Logarithmic risk preferences ($\alpha = 0$) are also a special case of (5.8) where

$$m_t = \frac{1}{\tilde{M}_{t+1}}$$

We see that in this case the IMRS moments are determined by the stochastic properties of the market real return, as in the static version of the CAPM, confirming the earlier claim that this class of models includes both the C-CAPM and the static CAPM as special cases. We might wish to determine the moments of the IMRS outside of such singular cases as these by allowing any "reasonable" specification of preferences in equation (5.8).

It is interesting to compare the functional forms of utility between equation (5.8) and equation (5.8.1). In equation (5.8.1) it is clear that the volatility of the gross consumption growth rate, and thus the theoretical IMRS from the model, is primarily a function of the CRRA parameter, so that to magnify the natural consumption growth volatility the level of risk aversion must be increased. As discussed in section one, previous studies have demonstrated that to mimic asset return volatility displayed in U.S. markets the risk parameter in equation (5.8.1) must assume an implausibly large value.

On the other hand, both the CRRA and EIS parameters in equation (5.8) affect the level of aggregate risk in the model. In particular, the parameter γ will assume large values for any given level of risk aversion as the value of ρ is reduced toward zero. This suggests that using equation (5.8) can generate large levels of aggregate risk for moderate levels of risk aversion if the level of aversion to intertemporal substitution is progressively reduced. This may prove useful for matching moments in an HJ test.

Model of Kandel and Stambaugh (1991)

Using the results from above, Kandel and Stambaugh (1991) derive a modified but essentially equivalent expression for the theoretical IMRS of a representative agent. Their model is discussed here because the authors raise some issues relevant to the results in sections four and five. The model assumes an infinitely-lived consumer who maximizes lifetime utility V which has the recursive structure

$$V_t = \left[c_t^{(\eta-1)/\eta} + \beta \left[E_t \{ \tilde{V}_{t+1}^{(1-\alpha)} \} \right]^{\frac{(\eta-1)}{\eta(1-\alpha)}} \right]^{\eta/(\eta-1)}$$

where $0<\beta<1$, $0<\alpha\neq1$, and $0<\eta\neq1$. In this formulation, β represents the rate of time preference when future utility is deterministic, α is the coefficient of relative risk aversion for atemporal gambles, η is the elasticity of intertemporal substitution, and $E_t\{V_{t+1}^{(1-\alpha)}\}$ denotes the certainty equivalent of utility at time t+1. The Euler equation associated with the joint maximization of the utility index with respect to real consumption and portfolio weights is shown to be

$$\beta^{\frac{\eta(\alpha-1)}{(1-\eta)}} E_t \left[\left(\frac{\tilde{c}_{t+1}}{c_t} \right)^{\frac{(1-\alpha)}{(1-\eta)}} \left(1 + \tilde{R}_{A,t+1,1} \right)^{\frac{(\alpha\eta-1)}{(1-\eta)}} \left(1 + \tilde{R}_{k,t+1,1} \right) \right] = 1$$

where $R_{A,t+1,1}$ denotes the one period real rate of return from time t to time t+1 on the consumer's optimal portfolio ("aggregate wealth") and $R_{k,t+1,1}$ denotes the one period real rate of return on any asset k. This recursive formulation implies that

$$m_t = \beta^{\frac{\eta(\alpha-1)}{(1-\eta)}} \left[\left(\frac{\tilde{c}_{t+1}}{c_t} \right)^{\frac{(1-\alpha)}{(1-\eta)}} \left(1 + \tilde{R}_{A,t+1,1} \right)^{\frac{(\alpha\eta-1)}{(1-\eta)}} \right]$$

which holds regardless of the information set available to the representative investor. Note that when $\alpha = 1/\eta$ we obtain

$$m_t = \beta \left[\left(\frac{\tilde{c}_{t+1}}{c_t} \right)^{-\alpha} \right]$$

Kandel and Stambaugh discuss this model in the context of the equity premium puzzle (see section 1). The authors agree with Weil (1989) that both the average risk-free interest rate and the equity premium fall with η because of higher asset demand in general, so that simply lowering η for a given value of α will not allow a resolution of the equity premium puzzle. The solution to the puzzle lies in increasing α for a given η which will raise the equity premium by reducing equity asset demand while simultaneously lower the risk-free interest rate by increasing demand for the risk-free asset through precautionary

saving. This occurs even with the special case of time-additive EU. Thus, Kandel and Stambaugh find that values of the CRRA as high as thirty are not unreasonable after examination of annual U.S. asset return data, a conclusion which mirrors Kocherlakota's (1990b, 1995) claims discussed earlier.

However, Kandel and Stambaugh do observe that the volatility of asset returns is inversely related to the value of η, and that with $\eta= 1/29$ their model gives an implied volatility equal to the results in Mehra and Prescott (1985). This is so because a lower η, which corresponds to a stronger preference for intertemporally smooth consumption, increases the volatility of wealth that is consistent with a given volatility of consumption, thus increasing the volatility of the theoretical IMRS. The authors also note that α plays a negligible role in driving asset return volatility - they observe that varying α between 0.5 and 29 produces virtually no effect on the implied volatility of asset returns in their model, a result that stands in contrast to non-GEU results.

Whether this ability of these GEU models to mimic the asset return volatility using consumption data by reducing the EIS parameter implies that these models are useful for explaining asset price movements remains to be seen.

Model of Kocherlakota (1995)

Kocherlakota (1995) comments on a GEU representation similar to the one used by Epstein and Zin (1991). The Euler equation from which the theoretical IMRS is drawn is

$$E_t \left[\beta^\gamma \left(\frac{\tilde{c}_{t+1}}{c_t} \right)^{-\gamma \rho} (\tilde{R}^m_{t+1})^{\gamma-1} \tilde{R}^b_{t+1} \right] = 1$$

where $\gamma \equiv (1-\alpha)(1-\rho)^{-1}$, $(1-\alpha)$ denotes the CRRA, $(1-\rho)^{-1}$ denotes the EIS, R^m denotes the gross real return to the optimal portfolio, and R^b denotes the gross real return to bonds. The Euler equation implies that

$$m_t = \left[\beta^\gamma \left(\frac{\tilde{c}_{t+1}}{c_t} \right)^{-\gamma \rho} (\tilde{R}_{t+1}^m)^{\gamma - 1} \right]$$

which holds regardless of the information set of the representative investor. Kocherlakota indicates the R^m is not observable and that a stock market proxy (such as the S&P 500) will understate the true level of diversification of the representative investor and potentially overstate the covariability of the IMRS with the proxy variable. Furthermore, he concludes that using such a stock proxy will lead to the "spurious" conclusion that GEU preferences can resolve the equity premium puzzle with "low" levels of risk aversion. This conclusion will be implicitly tested in this study, as will Kocherlakota's concern regarding the covariance issue.

Data

In addition to the use of a GEU preference specification, another feature of this study is its use of seasonally unadjusted consumption and price deflator data. This type of data is not commonly used in the consumption-based asset pricing literature, and it has received only limited attention for reasons not entirely clear. It seems that unadjusted data would be preferred for analysis as it captures actual consumption in each period at actual prices (who purchases anything at seasonally adjusted prices?), and therefore presents a more realistic idea of the actual environment in which investors allocate resources. Seasonal adjustment involves an artificial smoothing of the data which changes the true time series profile of a variable in often significant ways. Furthermore, an unadjusted series will have greater volatility than an adjusted one, which may be important to consider when attempting to explain asset price volatility using consumption data. The three data sets discussed here are consumption, Treasury bill return, and market return.

Consumption Data

Consumption data used in the models in section two was obtained from the commerce department's not seasonally adjusted data on aggregate quarterly consumer expenditures for nondurable goods and services for the period 1964:III through 1986:IV in the U.S. (which corresponds with the Treasury bill return data used in Figure 6 of Hansen and Jagannathan, 1991).[1] The omission of durable goods expenditures is reasonable here as they do not represent current consumption expenditures for the purposes of understanding asset market behavior, especially for short-term bonds and other securities. Gallant and Tauchen (1989) suggest that ignoring durable goods in this way does not represent a serious misspecification of utility.

The commerce department's consumption series measure nominal quantities, whereas the models in section two are all expressed in real terms. Price deflators for personal consumption expenditures are available only in seasonally adjusted form, but unadjusted implicit price deflators (IPD's) are available. I use the nondurable goods and services components of the not seasonally adjusted IPD to deflate the nominal consumption expenditures, thereby producing a quarterly series of real, not seasonally adjusted aggregate real consumption. Although 90 quarters of consumption and market return data were obtained for this study, only 89 quarters of data were actually used in the calculations due to the missing lead values required to compute the growth rates for 1986:IV.

However, the use of seasonally unadjusted consumption data to help explain asset price volatility does not eliminate other potential problems with using consumption series as measures of marginal utility, such as (i) omitting classes of consumption goods that

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[1] Aggregate, rather than per-capita, consumption data is used in an attempt to avoid artificially "smoothing" the data more than is necessary. This motivates the use of seasonally-unadjusted consumption data as well. An examination of the time-series properties of aggregate and per-capita consumption growth rates reveals this smoothing effect (although it appears to be quite small in magnitude relative to eliminating the seasonal component of consumption data - no doubt due to the stability of the population growth rate in the U.S.).

provide utility, creating a "missing variables" problem if preferences are not state-separable across goods (a similar problem arises regarding market return measurements), (ii) infrequent and "nonsynchronous" sampling of consumption expenditures, (iii) time aggregation, which artificially smoothes actual consumption series, (iv) publication lags, (v) measurement errors, and (vi) differing tax treatment between consumption goods and security returns (Ferson and Harvey, 1992, p.521).

Figure 5.1(a) illustrates the quarterly time series of seasonally unadjusted aggregate real consumption growth. A strong seasonal effect is observed as well as the absence of any apparent trend. The partial autocovariance plot in Figure 5.1(c) indicates a high positive autocorrelation at the seasonal lag, as well as high negative autocorrelation at the first and third period lags. As noted by Ferson and Harvey, these seasonally unadjusted consumption patterns differ significantly from seasonally adjusted consumption data frequently used in studies of consumption-based asset pricing. The seasonally adjusted data is smoothed by the U.S. government with the X-11 seasonal-adjustment program which takes weighted averages of past and (in revised data) future expenditures, and creates potential problems such as using future expenditures in current period data, which cannot provide utility in the current period, and inducing bias and erroneous inference due to spurious correlation between error terms and past values of variables in a model (Wallis, 1974). Compared with seasonally adjusted consumption data, seasonally unadjusted real consumption growth rates exhibit very different overall means and autocorrelation structure, as well as standard deviations which are an order of magnitude larger (Ferson and Harvey).

The autocorrelations in Figure 5.1(b) appear to decay toward zero at longer lags. The systematic sign reversal pattern with slow decay indicates a stationary but strongly seasonal time series; together with the lack of any trend in the observations, this suggests that the quarterly aggregate real consumption growth rate is a stationary process.

Average aggregate nondurable goods expenditures are high in the second and fourth quarters and usually fall in the first and third quarters, whereas average services
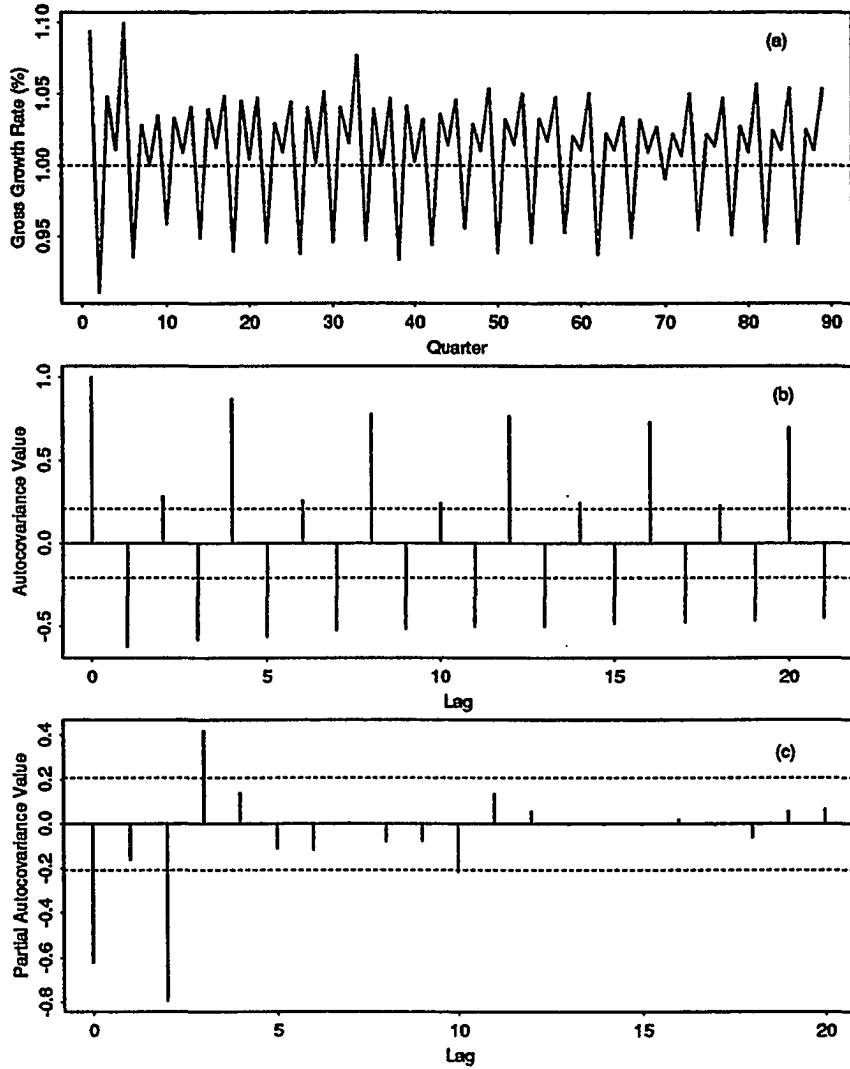
Figure 5.1. Time-series properties of the real aggregate, seasonally-unadjusted US nondurable goods and services consumption growth rate: 1964:III to 1986:III.

expenditures are highest in the first quarter and lowest in the second quarter. Aggregate services and nondurable goods expenditures are negatively correlated, reflecting the opposing seasonal patterns. However, because nondurables are by far the larger of the two expenditure components used as the consumption measure in this study, the sum behaves very much like the nondurables component.

Treasury Bill Return Data

The data on Treasury bill returns was obtained from the World Wide Web Virtual Library finance section and collected by the Investment SIG of the Capital PC Users Group. The data series there is reported in nominal values and represents secondary market averages of daily closing bid prices over one month intervals, with quarterly returns then computed for each month.

Real quarterly returns were obtained from this data set by averaging monthly data, calculating percentage changes between each quarter, and deflating the resulting nominal returns by the same IPD components used to deflate the consumption series. Figure 5.2(a) illustrates the quarterly time series of real Treasury bill returns over the relevant time period. As in the quarterly consumption real growth rate series we see a strong seasonal effect as well as the absence of any noticeable trend. The partial autocovariance plot in Figure 5.2(c) indicates a high positive autocorrelation at the seasonal lag as well as the first and second period lags.

Average real Treasury bill returns are high in the second and fourth quarters and fall in the first and third quarters. The seasonal component of the quarterly price deflator series (see Figures 5.3(a), (b), and (c)) drives the seasonal pattern observed in the real Treasury bill return series, as nominal quarterly returns exhibit little seasonal pattern, whereas both real and nominal quarterly aggregate consumption expenditure growth rates exhibit strong seasonal patterns. The similar time series pattern between real quarterly aggregate consumption growth rates and real quarterly Treasury bill returns suggests that
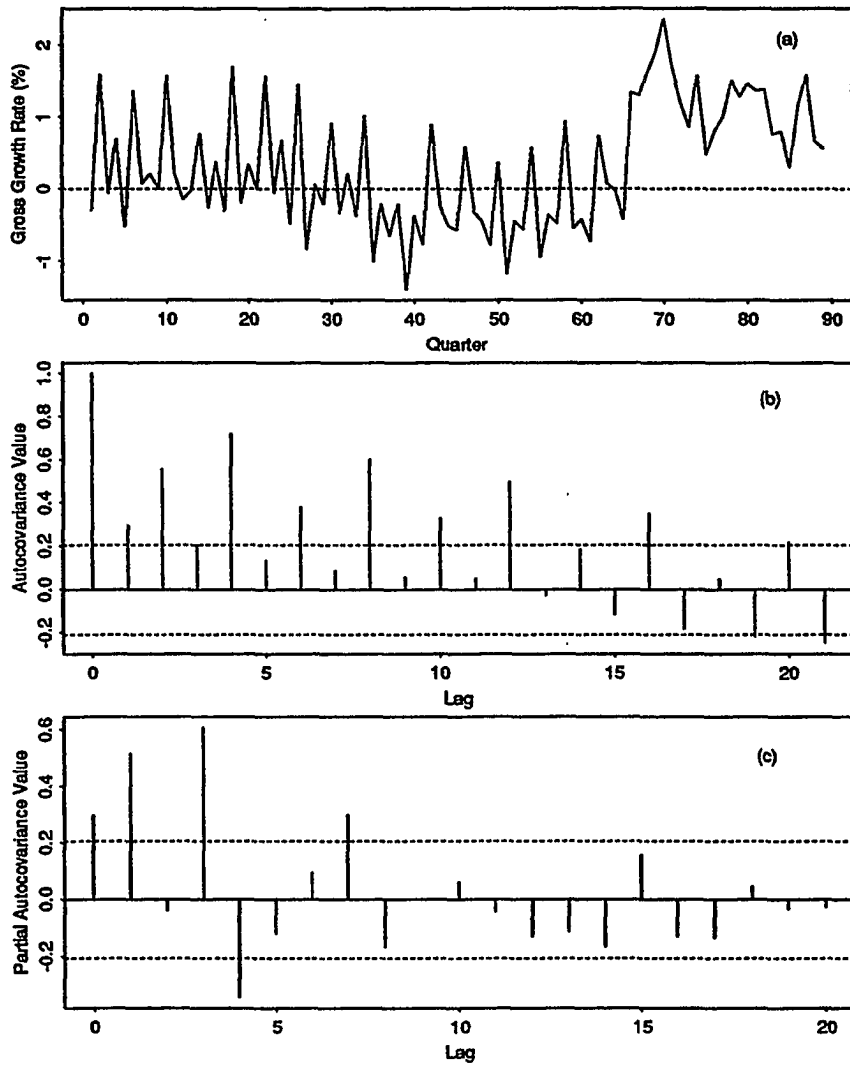
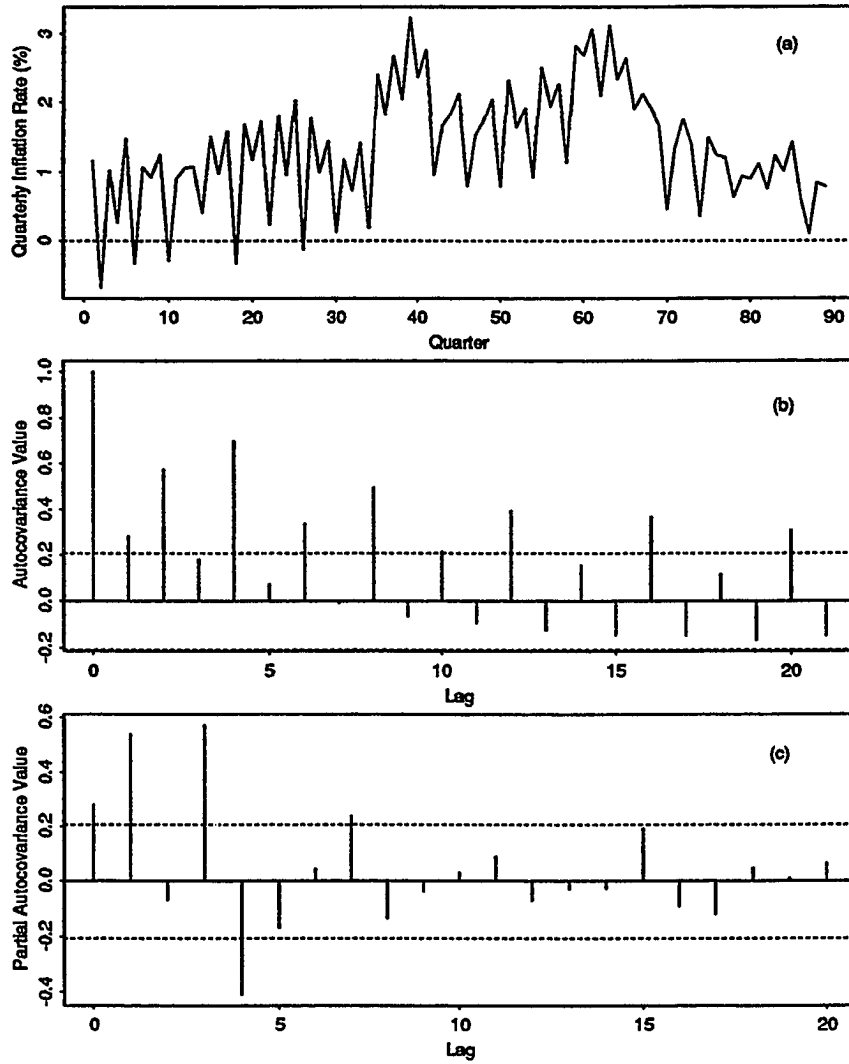Figure 5.2. Time-series properties of quarterly real returns on US Treasury bills: 1964:III to 1986:III.

Figure 5.3. Time-series properties of the quarterly inflation rate on nondurable goods and services in the US: 1964:III to 1986:III.

consumption-based models of asset prices may be particularly useful when analyzing the secondary market for U.S. Treasury bills (at least at the quarterly frequency).

Market Return Data

The market portfolio proxy used in the class of models in section two is the Standard and Poors (S&P) 500 stock index return, which was retrieved from the finance section of the World Wide Web Virtual Library. The data series there is reported in nominal values and represents the end-of-the-month index value for each month of the year.

Real quarterly returns were constructed in the same way as real Treasury bill returns. Figure 5.4(a) illustrates the quarterly time series of real S&P index returns over the relevant time period. A slight positive partial autocorrelation is detected at the first lag in Figure 5.4(c), but the autocovariance plot in Figure 5.4(b) and the absence of any trend indicate that this series is relatively stationary with no seasonal effects present in the data.

Estimation and Results

The analysis of the models was conducted, and the time series graphics were produced, on Splus Version 3.1 Release 1. Computations were done over the range of relative risk aversion values ranging from zero to thirteen in unit intervals, and over the range of EIS values ranging from zero to two in intervals of 0.05. For each pair of preference parameters (CRRA, EIS) a series of 89 IMRS ("m") values were produced, one for each quarter over the time period 1964:III through 1986:III, using the consumption growth and market return data discussed in section three. The first two moments (i.e. the $\{E(m), \sigma(m)\}$-pairs) of each theoretical IMRS series were then calculated, with selected
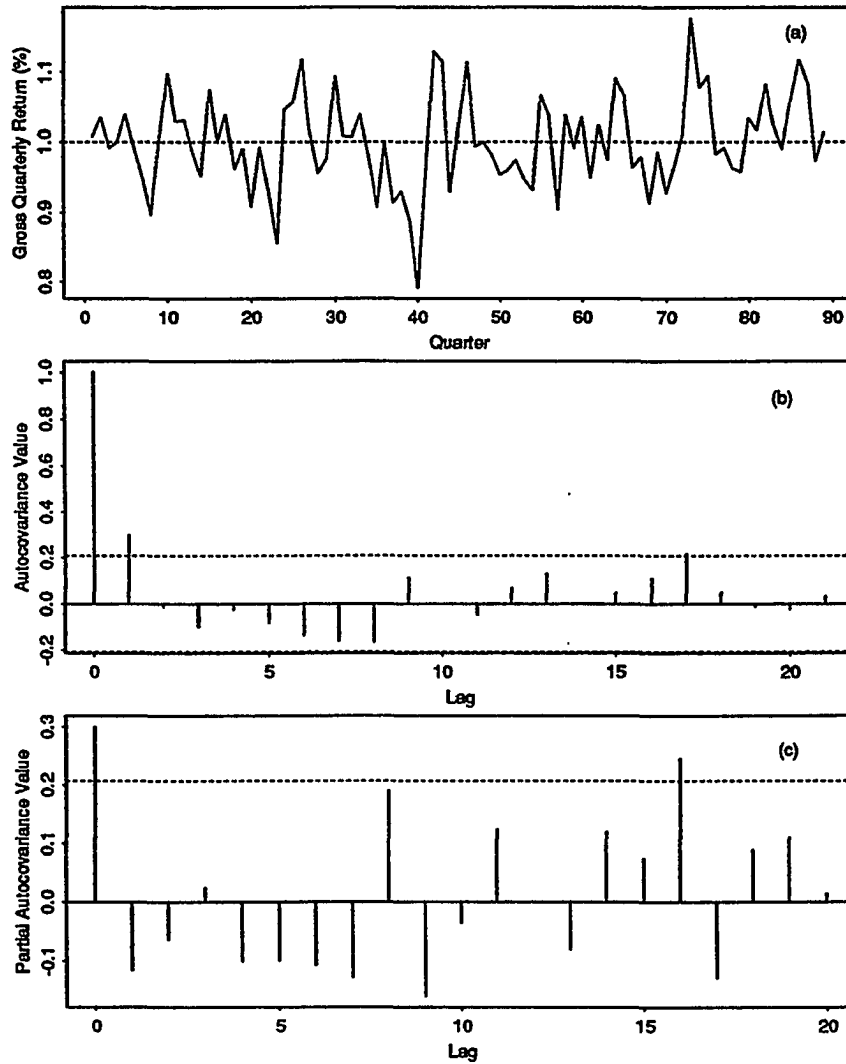
Figure 5.4. Time-series properties of quarterly real returns on the Standard & Poor's 500 stock index: 1964:III to 1986:III.

moment data appearing in Table 1.

A general pattern emerges, which is visually displayed in Figure 5.5. The figure shows the general path of the IMRS moments {E(m), σ(m)} implied by the GEU models as the level of risk aversion is increased for a given level of the EIS. The path is meant to be representative of the results obtained over the entire preference parameter space examined. Also plotted is the HJ lower bound constructed using Treasury bill real return data. Although the volatility of the theoretical IMRS values calculated from the GEU models can easily match the volatility requirements of the HJ frontier, it is clear from both the table and Figure 5.5 that the GEU model will not, by itself, produce moment restrictions that lie within the HJ bounds. As ρ is the key parameter inducing high volatility in these models, we see that Kocherlakota's (1995) concern about covariability problems between the IMRS and the market proxy is not troublesome. High theoretical IMRS volatility, produced particularly when the EIS parameter is increased toward unity (contrary to the findings of Kandel and Stambaugh, 1991), occurs only with first moment which are significantly greater than one. This implies that to generate volatility sufficient to match the HJ bounds, the theoretical models require that the representative individual be willing to pay more to acquire any particular asset than the unconditionally expected discounted return on the asset over the holding period. When more realistic IMRS mean values (i.e. E(m)<1) are generated in these models, the standard deviations are always significantly below those of the HJ bounds. Over the range of preference parameter examined, a convex outer hull of moment pairs is plotted in Figure 5.6 for the GEU model and compared to the moment pairs produced by imposing the restriction that α = ρ. Two interesting results emerge: (i) the general model is able to account for only two-thirds of the return volatility in the Treasury bill markets, and (ii) the gain from generalizing the C-CAPM to the Epstein-Zin model appears to be negligible for the functional forms assumed and the data sets examined. This second result stands in contrast to the more favorable evaluations of the Epstein-Zin model found in the literature which use parametric

Table 1. Theoretical IMRS Moments from GEU Model using Consumption and Market Return Data.[1]

EIS	CRRA						
	0	1	2	3	4	5	6
0.10	0.99	0.99	0.98	0.97	0.97	0.96	0.97
	0.08	0.06	0.07	0.10	0.15	0.19	0.24
0.20	1.00	0.99	0.98	0.97	0.97	0.97	0.97
	0.09	0.06	0.07	0.11	0.16	0.22	0.28
0.50	1.01	0.99	0.97	0.97	0.98	1.01	1.05
	0.15	0.06	0.08	0.19	0.31	0.44	0.61
0.67	1.03	0.99	0.98	1.00	1.06	1.17	1.34
	0.24	0.06	0.15	0.35	0.62	0.97	1.46
0.83	1.13	0.99	1.02	1.27	1.90	3.32	6.48
	0.53	0.06	0.41	1.20	2.95	7.03	16.84
0.91	1.48	0.99	1.25	3.09	11.76	54.44	275.00
	1.45	0.06	1.13	6.29	34.36	195.00	1142.00
0.95	3.81	0.99	2.98	49.00	1248.00	36583.00	1157909.00
	10.68	0.06	5.95	74.00	5720.00	199264.00	7156906.00
1.05	2.80	0.99	3.65	102.00	6277.00	444567.00	32619949.00
	5.36	0.06	10.52	734.00	54417.00	4064042.00	303877945.00
1.25	1.02	0.99	1.10	1.39	2.03	3.44	6.76
	0.34	0.06	0.49	1.37	3.70	10.30	29.41

[1] Data appear as moment pairs with $E(m)$ above $\sigma(m)$ values. $E(c_{t+1}/c_t) = 1.0101$ and $E(M_t) = 1.0161$ over the relevant time period 1964:III to 1986:III. Higher values of the EIS (above 1.25) do not improve the performance of the models from what is shown in the table.
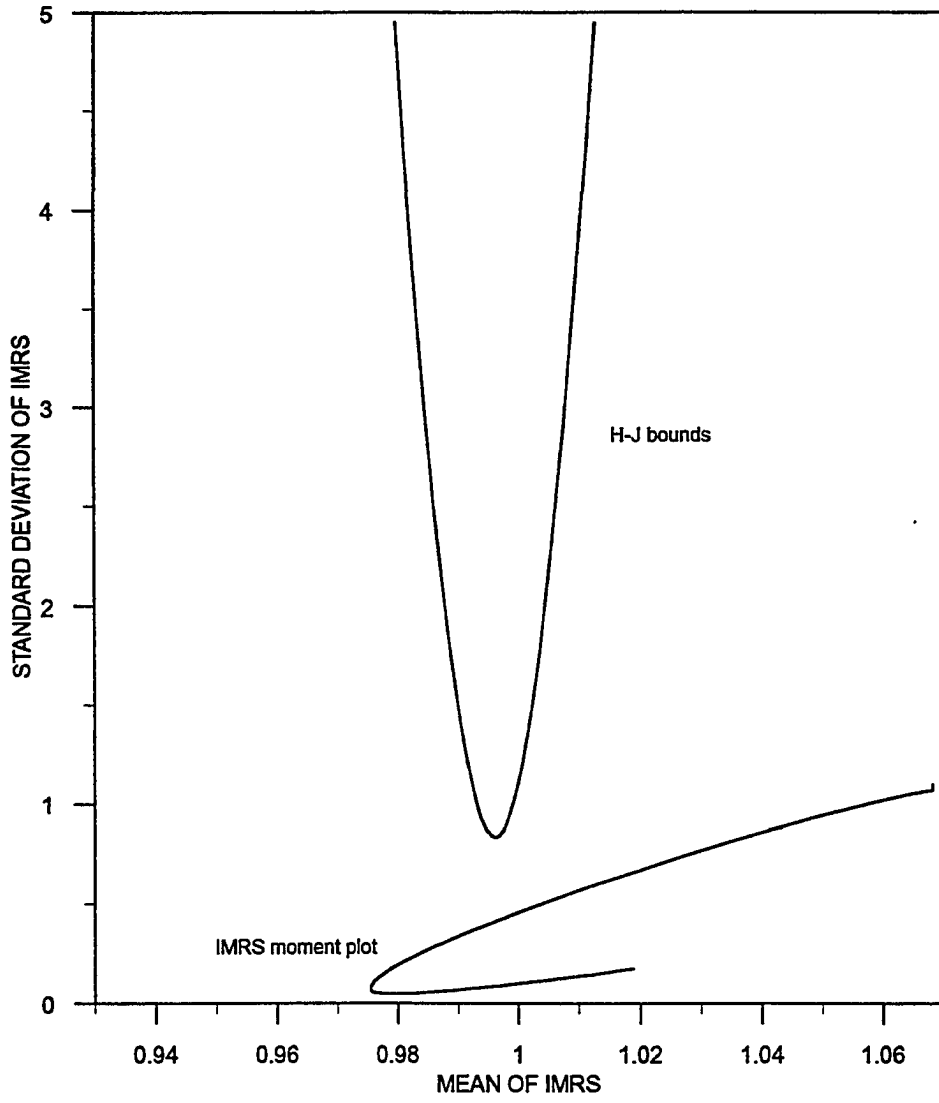
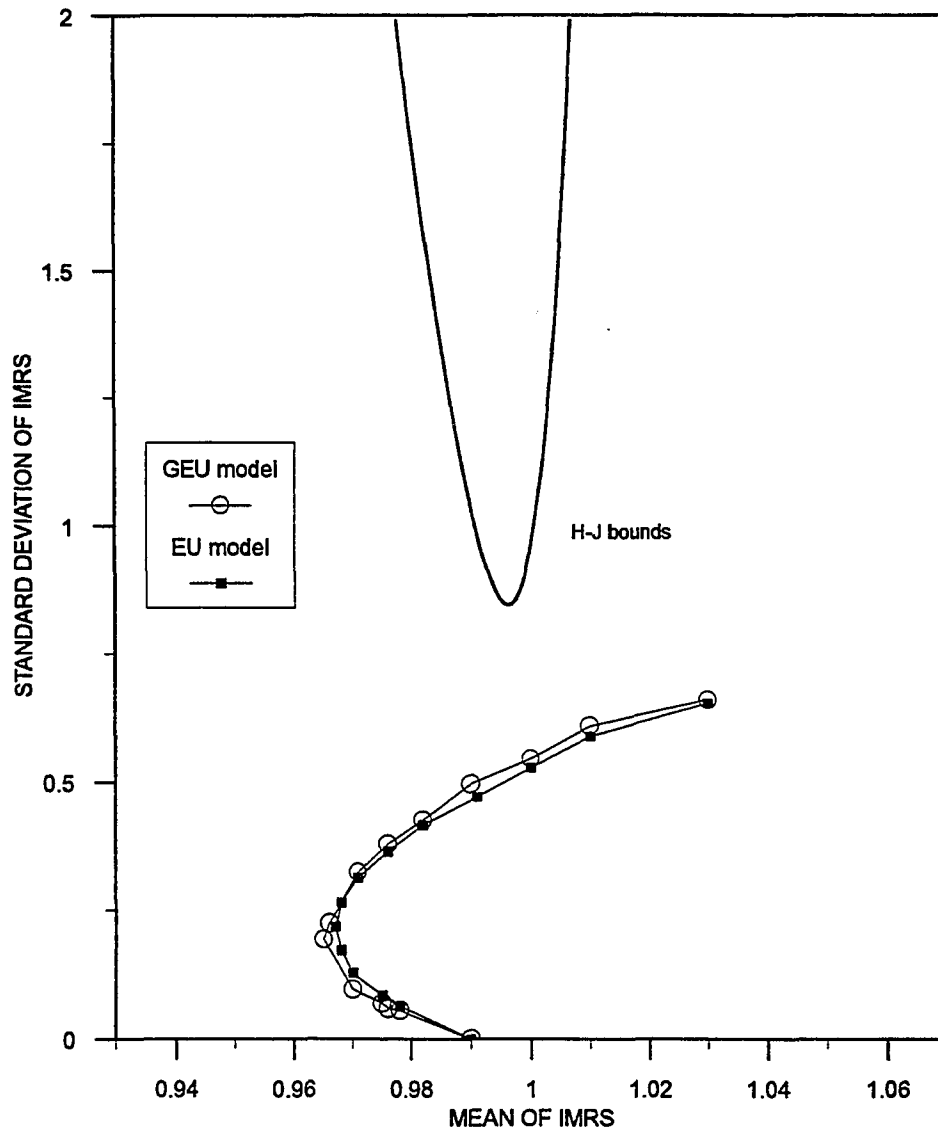Figure 5.5. Hansen-Jagannathan (H-J) bound and moment pattern of GEU model.

Figure 5.6. H-J bound with moment pairs for one-factor EU IMRS plotted against the convex hull of moment pairs for two-factor Epstein-Zin IMRS over the range CRRA ∈ {0,13}.

(typically GMM) tests.

This inability of the GEU model to match the volatility of Treasury bill real returns over a reasonable mean value range of the IMRS is significant since Treasury bill real returns have historically exhibited relatively low volatility compared to real returns on most other assets in the U.S. If these GEU models fails to match moments using Treasury bill data, it seems unlikely that they will fare better when the HJ bounds are constructed using data from more volatile asset markets. A tentative observation from these results is that this type of GEU modeling, even when used with seasonally unadjusted consumption and market return data, does not appear well-suited to resolve the equity premium puzzle.

Discussion and Conclusions

Gregory and Smith (1992) and Burnside (1994) indicate that sampling error may result in the estimated HJ lower bounds on the standard deviation of the IMRS to be too high compared to the actual variance. This is important here since the HJ bounds used to test this class of GEU models were constructed using four different assets, increasing the likelihood, due to sampling variability, of type I errors in the test procedure.

Also, as Hansen and Jagannathan point out, the volatility bounds they construct for quarterly holding period returns on Treasury bills may not be entirely reliable since the Treasury bill prices used to construct the bounds were drawn from low volume secondary markets where the bid-ask spread can be significant (particularly in the case of 3-month bills, which are normally held to maturity). In addition, these types of bonds may have value to an investor beyond the real return. Short-term Treasury bills are often held for liquidity purposes as cash-substitutes for some types of transactions - this liquidity "return" is not incorporated into the HJ bound used appearing in Figures 5.5 and 5.6. As such, the measured real returns may understate the actual value of the bonds to an investor (perhaps explaining a significant part of the risk-free rate puzzle). This is relevant since the large

volatility bounds of the IMRS using Treasury bill return data are due to the expected short-term gain associated with holding longer term (6-, 9-, and 12-month) notes being larger than the increase in the standard deviation. Thus, a distortion in the magnitude of the bounds may be occurring by abstracting away from these liquidity services which would only be corrected by introducing the role of money and price level determination to fit observed Treasury bill return data.

However, it seems unlikely that this omission, by itself, can account for the poor performance of the theoretical GEU or EU models, and it is interesting to see how badly the GEU model falls short of matching asset market moment restrictions on the IMRS of a representative investor given its extensive use in the intertemporal asset pricing literature (see section one). It is clear that a simple separation of the CRRA from the EIS will not generate sufficient aggregate risk in the theoretical model to explain asset price volatility unless additional implausible market behavior is assumed, even in a market as tranquil as the U.S. Treasury bill market.

Mathematically, the theoretical model fails because the mean of the gross real market return over the relevant time period is 1.0161, whereas the mean of the consumption growth series is 1.001; thus, as the volatility of the IMRS series as calculated in the GEU model is increased by simultaneously increasing both the CRRA and EIS preference parameters, the IMRS mean is likewise increased, and it is clear that the "market return" effect on the mean quickly dominates the "consumption" effect. This can be seen clearly using equation (5.8), where

$$\left[\beta \left(\frac{\tilde{c}_{t+1}}{c_t} \right)^{\rho-1} \right]^{\gamma}$$

decreases as $|\alpha|$ increases and $|\rho|$ decreases, whereas

$$(\tilde{M}_{t+1})^{\gamma-1}$$

increases. Economically, the GEU preference specification fails the HJ volatility bound test because, as mentioned above, they lead to the conclusion that the level of aggregate

risk in the economy necessary to explain the volatility in the asset markets will produce investment behavior such that asset prices actually exceed the discounted unconditional expectation of their return, a behavioral implication that has not been observed in actual U.S. asset markets over any extended period of time.

Although the excess volatility puzzle for theoretical modeling does not appear to be an artifact of the consumption smoothing implied by seasonal adjustment, the difficulty may well be inherent in the use of aggregate consumption data. These data are measured with error and are time-aggregated, which can have serious consequences for asset pricing relationships (Wheatley, 1988 and Heaton, 1995). More fundamentally, the consumption of asset market participants may be poorly proxied by aggregate consumption. The consumption of stock- and bond-holders differs significantly from the consumption of other population subgroups (Campbell, 1993). However, obtaining consumption data on this subset of the population would be exceedingly difficult.

Epstein and Melino (1995) suggest that it is the particular functional form of the GEU utility index, rather than the recursive modeling itself, that is producing the poor fit with the data. Using revealed preference analysis of asset pricing under a recursive utility framework, they observe that the generalization from expected utility to recursive utility "contributes substantially" to the resolution of the equity premium puzzle.

With regard to functional form, Epstein and Melino note that it is still an unresolved empirical question whether relaxing time-separability (e.g. introducing durable goods, habit formation and local substitution behavior) or state-separability inherent in traditional consumption-based asset pricing models is more useful for explaining and organizing the observed movements of consumption and asset returns (see Gallant and Tauchen, 1989, Gallant et al., 1990, Constantinides, 1990, Hansen and Jagannathan, 1991, Cochrane and Hansen, 1992, and Heaton, 1995). However, they do note that relaxation of time separability does not permit a separation between the CRRA and EIS preference parameters, when such a separation appears important for purposes of understanding saving and investment behavior. Hansen and Jagannathan examine time-nonseparability in

an EU framework and find that habit persistence and intertemporal consumption complementarity produce IMRS moments that satisfy HJ bounds only when the CRRA $\approx$ 14 (although time-separable EU models imply still higher CRRA values). Gallant et.al use a seminonparametric methodology suggested by Gallant and Tauchen to estimate the conditional distributions of a vector of monthly asset payoffs as well as to calculate both the conditional and unconditional volatility frontiers for the IMRS. Using a time-nonseparable EU framework their results are similar to those of Hansen and Jagannathan. The recent paper by Heaton uses Simulated Method of Moments (SMM) instead of GMM to show that temporal disaggregation (constructing weekly consumption data) and time-nonseparability in the form of local durability/substitutability of consumption within a period of four months with habit persistence/complementarity of consumption occurring over longer periods of time can significantly improve the performance of asset pricing models in terms of the HJ bounds test. However, as discussed above, relaxing time-separability does not allow any of these researchers to separately parameterize time and risk preferences, as would seem desirable.

Other candidate functional forms and extensions of the GEU framework which may perform better than the basic model above include: (i) introducing asymmetric market fundamentals (Hung, 1994), (ii) broadening the proxy for the market portfolio beyond the S&P 500 index (Wheatley, 1986) which, as Kocherlakota (1990, p. 303) points out, will reduce the estimated variance of the market portfolio return and thus increase estimates of the CRRA in parametric work, which typically use grossly inaccurate measures of wealth, (iii) introducing market friction (Lucas, 1994 and He and Modest, 1995), (iv) introducing incomplete market structures (Mankiw, 1986, Aiyagari and Gertler, 1991, Marcet and Singleton, 1991, Weil, 1992, Telmer, 1993, Constantinides and Duffie, 1994, and Heaton and Lucas, 1995), (v) reducing the degree of time averaging in the data (Heaton, 1995), (vi) using a first-order risk aversion specification to amplify the effects of small changes in risk on portfolio activity, and (vii) introducing money. To my knowledge, there are no

published papers which introduce both GEU preferences and money in a model of asset pricing.

Kocherlakota (1995, p. 17) suggests that the three GEU models examined in this study should "lead one to the spurious conclusion that GEU preferences can resolve the equity premium puzzle with low levels of risk aversion", which the C-CAPM fails to do. The testing performed here indicates that even a "spurious" resolution is not likely to occur due to the similar performances of the generalized model and the C-CAPM.

CHAPTER 6: CONCLUSIONS

The freedom to independently specify the functional forms representing preferences within generalized expected utility is an important contribution to stochastic intertemporal modeling in economics. It can enhance the theoretical understanding of a wide variety of economic topics ranging from precautionary saving to Ricardian equivalence to the permanent income hypothesis, as the work and cites in the text suggest.

With regard to intertemporal factor allocations and the use of them as a hedge against income uncertainty, we saw the importance of clearly distinguishing between risk attitude and intertemporal preferences in chapters three and four.

The simple two period model in chapter three demonstrated that intertemporal preferences are irrelevant in determining factor allocations when pure endowment income risk is present and when (i) intertemporal preferences are assumed homothetic and (ii) capital markets are assumed perfect in the sense that only one interest rate prevails in the economy.[1]

However, we also saw in chapter three that intertemporal preferences are critical in determining factor allocations when pure capital income risk is present and assumptions (i) and (ii) above hold.

Regarding pure wage income risk and intertemporal factor allocations, it was shown in chapter three that intertemporal preference plays a more crucial role than risk attitude in predicting behavior, particularly if aversion to intertemporal substitution is high.

This asymmetric treatment of preferences disappears within an isoelastic expected utility framework, as chapter four demonstrated. Due to the axiomatic restrictions on the choice of functional forms representing preferences, expected utility does not distinguish

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[1] As noted in chapters three and four, an interesting extension of the GEU results would be to consider a more realistic situation where the borrowing interest rate exceeds the lending rate. It is hypothesized that this extension would reduce the importance of intertemporal preferences and enhance the role of risk aversion in determining intertemporal factor allocation decisions under any type of income risk.

between the two critical preference concepts in an intertemporal setting. The result is a perfectly symmetric treatment of time and risk preferences that is easy to misinterpret and can lead to the erroneous conclusion that only risk attitude matters for decision-making in a stochastic intertemporal environment, as some classic results in saving theory reveal.

Multivariate extensions of the three univariate cases in the generalized model of chapter three show that the asymmetric treatment of preferences is generally reproduced and follows the same pattern (given the model assumptions): risk attitude remains the crucial parameter to understanding intertemporal factor allocations under income risk when the intertemporal price of consumption and leisure remain constant, whereas intertemporal preference is the critical parameter in predicting factor allocations over time when income risk is present and if either the intertemporal price of consumption or leisure is affected by it.

In addition, the correlation of capital returns with either wage income or endowment income is shown to be important for understanding intertemporal behavior. The correlation terms that are produced in the generalized model may be either positive or negative so that the presence of two stochastic income sources may either magnify, dampen, or even reverse the comparative static implications derived in the three univariate cases, depending on the strength of the income variance and covariance expressions.

In particular, for any given level of risk aversion, a moderate level of aversion to intertemporal substitution combined with a low level of current saving may result in a situation where higher capital return variance generates even lower saving in the univariate case but produces higher saving in a multivariate case if capital returns are strongly correlated with either stochastic wage income or stochastic endowment income.

These income correlation considerations point out that an individual who is averse to both risk and intertemporal substitution will be sensitive to the riskiness of total future income when considering intertemporal factor allocation choices.

Thus, a separation of the two preference concepts produces a clearer theoretical explanation of the motivations behind the use of factor supplies as a hedge against income risk in a two period framework.[2]

However, the results of chapter five demonstrate that this clean separation of preferences does not necessarily produce useful advantages in the modeling of asset return behavior, thus supporting the intuition of Weil (1989, 1990) and Kocherlakota (1995).

Using the nonparametric diagnostic tool devised by Hansen and Jagannathan (1991), it is shown in chapter five that the use of a recursive class of preferences which extends the model of chapter three to an infinite-horizon framework, and thus delivers the desired separation of time and risk preferences, fails to generate sufficient aggregate risk within the C-CAPM to adequately explain the volatile time profile of recent U.S. Treasury bill return data. In addition, it is shown that the two-factor Epstein-Zin model fails to perform significantly better than the standard one-factor C-CAPM model.

These result suggests that this class of preference representation may have limited value for asset pricing applications, despite its frequent use in both the theoretical and empirical literature.

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[2] Possible extensions include expanding the time-horizon beyond two periods, although tractable results may be difficult to derive. The use of a continuous-time framework may reduce this difficulty - see Merton (1990).

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APPENDIX A: CHAPTER 3 DERIVATIONS

Here we will derive the results discussed in chapter 3. The univariate results can be obtained by setting the unwanted variance and covariance terms equal to zero and replacing mean values with certainty values. The arguments of the two indirect utility functions are dropped for ease of notation.

THE CERTAINTY CASE

The objective of this section is to derive the form of the indirect utility function which will be used in all of the uncertainty cases as well as the certainty case. The primitive utility function is Cobb-Douglas so that the objective function is to

Max. $b\ln(c_t) + (1-b)\ln(1-n_t)$

$(c_t,(1-n_t))$

s.t. $y_t \geq c_t + w_t(1-n_t)$

The Lagrangian for this problem appears as

$\mathcal{L} = b\ln(c_t) + (1-b)\ln(1-n_t) + \mu[y_t - c_t - w_t(1-n_t)]$

The first order conditions for an interior solution are:

$c_t : \dfrac{b}{c_t} - \mu = 0$

$(1-n_t) : \dfrac{1-b}{(1-n_t)} - \mu w_t = 0$

which can be combined to obtain

$\dfrac{b}{c_t} = \dfrac{1-b}{(w_t(1-n_t))}$

Cross-multiply and we obtain

$b(c_t + w_t(1-n_t)) = c_t$

which, using the budget constraint, reveals the Marshallian demand for the period-t consumption good to be

$(c_t)^* = by_t$

Substituting $(c_t)^*$ into the budget constraint and we obtain the Marshallian demand for period-t leisure

$$(1-n_t)^* = \frac{(1-b)y_t}{w_t}$$

Finally we substitute the two Marshallian demands into the utility function itself to obtain the indirect utility function which specifies the maximum utility the representative agent can acheive given market prices and income

$v_t(c_t,(1-n_t)) = Aw_t^{b-1} y_t$

where $A \equiv (b^b (1-b^{1-b}))$, which is a constant.

The reduced optimization problem in the case of certainty will be as follows:

$$\underset{s}{\text{Max}} \quad -\frac{1}{\delta_1} [(v_1(c_1,(1-n_1)))^{-\delta_1} + \beta(v_2(c_2,(1-n_2)))^{-\delta_1}]$$

s.t. $v_1 = Aw_1^{b-1} y_1$

$\quad v_2 = Aw_2^{b-1} y_2$

$\quad y_1 = e_1 + w_1 - s$

$\quad y_2 = e_2 + w_2 + s(1+r)$

The first order condition will be of the form

$$\frac{\partial u(v_1,v_2)}{\partial s} = -\frac{1}{\delta_1} \left[\frac{\partial [(v_1)^{-\delta_1}]}{\partial s} + \beta \frac{\partial [(v_2)^{-\delta_1}]}{\partial s} \right] = 0$$

We can solve the first order condition in steps

$$\frac{\partial [(v_1)^{-\delta_1}]}{\partial s} = -\delta_1(v_1)^{-(\delta_1+1)} \left(\frac{\partial (v_1)}{\partial s} \right)$$

$$= -\delta_1(v_1)^{-(\delta_1+1)}[-Aw_1^{b-1}] = A\delta_1 w_1^{b-1}(v_1)^{-(\delta_1+1)}$$

$$\frac{\partial\ [(v_2)^{-\delta_1}]}{\partial\ s} = -\delta_1(v_2)^{-(\delta_1+1)}(\frac{\partial\ (v_2)}{\partial\ s})$$

$$= -\delta_1(v_2)^{-(\delta_1+1)}[(1+r)Aw_2^{b-1}] = -A\delta_1(1+r)w_2^{b-1}(v_2)^{-(\delta_1+1)}$$

Combining these results we obtain the reduced-form first order condition

$$\frac{\partial\ u(v_1,v_2)}{\partial\ s} = -\frac{1}{\delta_1}[A\delta_1 w_1^{b-1}(v_1)^{-(\delta_1+1)} - \beta A\delta_1(1+r)w_2^{b-1}(v_2)^{-(\delta_1+1)}] = 0$$

or rewriting we get

$$\left[\frac{v_1}{v_2}\right]^{-(\delta_1+1)} = \beta\ (1+r)\left(\frac{w_2}{w_1}\right)^{b-1}$$

This is the certainty version of the reduced-form first order condition against which all uncertainty versions can be compared. This first order condition defines an implicit function "F" such that

$$F = \left(\frac{v_1}{v_2}\right)^{-(\delta_1+1)} - \beta\ (1+r)\left(\frac{w_2}{w_1}\right)^{b-1} = 0$$

Implicitly differentiating the function we can derive comparative static results showing how saving behavior is modified following an income shock:

$$\frac{\partial\ s}{\partial\ e_1} = -\left(\frac{\frac{\partial\ F}{\partial\ e_1}}{\frac{\partial\ F}{\partial\ s}}\right) = \frac{v_2 w_1^{b-1}}{v_2 w_1^{b-1} + (1+r)v_1 w_2^{b-1}} > 0 \qquad \forall\, b\ \text{and}\ \forall\ \delta_1 > -1$$

$$\frac{\partial\ s}{\partial\ e_2} = -\left(\frac{\frac{\partial\ F}{\partial\ e_2}}{\frac{\partial\ F}{\partial\ s}}\right) = \frac{-v_1 w_2^{b-1}}{v_2 w_1^{b-1} + (1+r)v_1 w_2^{b-1}} < 0 \qquad \forall\, b\ \text{and}\ \forall\ \delta_1 > -1$$

$$\frac{\partial\ s}{\partial\ w_1} = -\left(\frac{\frac{\partial\ F}{\partial\ w_1}}{\frac{\partial\ F}{\partial\ s}}\right) = \frac{(\delta_1+1)\left(\frac{v_1}{v_2}\right)^{-(\delta_1+2)} v_2^{-1}Aw_1^{b-1} - \beta\ (1+r)(b-1)w_1^{-b}w_2^{b-1}}{(\delta_1+1)\left(\frac{v_1}{v_2}\right)^{-(\delta_1+2)} v_2^{-2}A\ (v_2 w_1^{b-1} + (1+r)v_1 w_2^{b-1})}$$

$$> 0 \qquad \forall\, b \le 1 \text{ and } \forall\, \delta_1 > -1$$

$$\frac{\partial s}{\partial w_2} = -\left(\frac{\dfrac{\partial F}{\partial w_2}}{\dfrac{\partial F}{\partial s}}\right) = \frac{\beta\,(1+r)(b-1)w_1^{1-b}w_2^{b-2} - (\delta_1+1)\left(\dfrac{v_1}{v_2}\right)^{-(\delta_1+2)} v_2^{-2}v_1 A w_2^{b-1}}{(\delta_1+1)\left(\dfrac{v_1}{v_2}\right)^{-(\delta_1+2)} v_2^{-2} A(v_2 w_1^{b-1} + (1+r)v_1 w_2^{b-1})}$$

$$< 0 \qquad \forall\, b \le 1 \text{ and } \forall\, \delta_1 > -1$$

$$\frac{\partial s}{\partial r} = -\left(\frac{\dfrac{\partial F}{\partial r}}{\dfrac{\partial F}{\partial s}}\right) = \frac{\beta\left(\dfrac{w_2}{w_1}\right)^{b-1} - (\delta_1+1)\left(\dfrac{v_1}{v_2}\right)^{-(\delta_1+2)} v_2^{-2}v_1 A s w_2^{b-1}}{(\delta_1+1)\left(\dfrac{v_1}{v_2}\right)^{-(\delta_1+2)} v_2^{-2} A(v_2 w_1^{b-1} + (1+r)v_1 w_2^{b-1})}$$

STOCHASTIC INCOME

For the case of joint endowment, wage, and capital income uncertainty the reduced optimization problem is

$$\underset{s}{\text{Max}} \quad -\frac{1}{\delta_1}[(v_1)^{-\delta_1} + \beta \, (v^e(\tilde{v}_2))^{-\delta_1}] \tag{3.1}$$

s.t.

$$v_1 = Aw_1^{b-1} y_1 = Aw_1^{b-1}(e_1 + w_1 - s) \tag{3.2}$$

$$v^e(\tilde{v}_2) = V^{-1} E \, V(\tilde{v}_2) \tag{3.3}$$

$$V(\tilde{v}_2) \equiv -\exp(-a\tilde{v}_2)$$

$$\tilde{v}_2 = A\tilde{w}_2^{b-1}\tilde{y}_2 = A\tilde{w}_2^{b-1}(\tilde{e}_2 + \tilde{w}_2 + s(1+\tilde{r}))$$

$$\begin{bmatrix} \tilde{e}_2 \\ \tilde{w}_2 \\ (1+\tilde{r}) \end{bmatrix} \sim \left(\begin{bmatrix} \mu_e \\ \mu_w \\ (1+\mu_r) \end{bmatrix}, \begin{bmatrix} \sigma_e^2 & \rho_{ew}\sigma_e\sigma_w & \rho_{er}\sigma_e\sigma_r \\ \rho_{ew}\sigma_e\sigma_w & \sigma_w^2 & \rho_{wr}\sigma_w\sigma_r \\ \rho_{er}\sigma_e\sigma_r & \rho_{wr}\sigma_w\sigma_r & \sigma_r^2 \end{bmatrix} \right)$$

Taking the expectation of the second order multivariate Taylor approximation of $V(\tilde{v}_2)$ around the point $(\mu_e, \mu_w, (1+\mu_r))$ we obtain

$$E[V(\tilde{v}_2)] \cong -\exp(-a\overline{v}_2) - \frac{1}{2}\left(aA\mu_w^{b-1}\right)^2 \exp(-a\overline{v}_2)\sigma_e^2 - \frac{1}{2}a\left(aG_1^2 - G_2\right)\exp(-a\overline{v}_2)\sigma_w^2$$

$$-\frac{1}{2}\left(aA\mu_w^{b-1}s\right)^2 \exp(-a\overline{v}_2)\sigma_r^2 - aA\mu_w^{b-1}\left(aG_1 - (b-1)\mu_w^{-1}\right)\exp(-a\overline{v}_2)\sigma_{ew}$$

$$-asA\mu_w^{b-1}\left(aG_1 - (b-1)\mu_w^{-1}\right)\exp(-a\overline{v}_2)\sigma_{wr} - \left(aA\mu_w^{b-1}\right)^2 s \exp(-a\overline{v}_2)\sigma_{er}$$

$$\cong -\exp(-av^e(\tilde{v}_2)) \qquad \text{(by equation 3.3)}$$

where

$$\overline{v}_2 \equiv A\mu_w^{b-1} (\mu_e + \mu_w + s(1+\mu_r))$$

$$G_1 \equiv \frac{\partial \overline{v}_2}{\partial \mu_w} = A\mu_w^{b-2} (b\mu_w - (1-b)(\mu_e + s(1+\mu_r)))$$

$$G_2 \equiv \frac{\partial^2 \overline{v}_2}{\partial \mu_w^2} = (b-1)A\mu_w^{b-3} (b\mu_w - (2-b)(\mu_e + s(1+\mu_r)))$$

Multiplying terms by

$$-\exp(a\overline{v}_2)$$

and taking natural logarithms we have after rearranging terms

$$v^e(\overline{v}_2) \cong \overline{v}_2 - a^{-1} \ln\Big(1 + \frac{1}{2}\big(aA\mu_w^{b-1}\big)^2 \sigma_e^2 + \frac{1}{2}a \big(aG_1^2 - G_2\big)\sigma_w^2 + \frac{1}{2}\big(aA\mu_w^{b-1}s\big)^2 \sigma_r^2$$

$$+ aA\mu_w^{b-1} \big(aG_1 + (1-b)\mu_w^{-1}\big)\sigma_{ew} + asA\mu_w^{b-1} \big(aG_1 + (1-b)\mu_w^{-1}\big)\sigma_{wr}$$

$$+ \big(aA\mu_w^{b-1}\big)^2 s\,\sigma_{er} \Big) \tag{3.4}$$

The reduced optimization problem can be written as

$$\underset{s}{\text{Max}} \quad -\frac{1}{\delta_1}[(v_1)^{-\delta_1} + \beta \, (v^e(\overline{v}_2))^{-\delta_1}]$$

s.t. (3.2) and (3.4) hold.

After some simplification the reduced form first order condition is

$$\left(\frac{v_1}{v^e(\overline{v}_2)}\right)^{-(\delta_1+1)} = \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} \Big[(1+\mu_r)+ K_w\sigma_w^2 + K_r\sigma_r^2 + K_{ew}\sigma_{ew} + K_{wr}\sigma_{wr} + K_{er}\sigma_{er}\Big]$$

where

$$K_w \equiv \frac{1}{2}(b-1)\mu_w^{-1}\big(1+\mu_r\big)\big((b-2)\mu_w^{-1} - 2aG_1\big)J^{-1}$$

$$K_r \equiv -asA\mu_w^{b-1} J^{-1}$$

$$K_{ew} \equiv a(1-b)A\mu_w^{b-2} J^{-1}$$

$$K_{wr} \equiv \Big((b-1)\mu_w^{-1} - a\big(G_1 + (b-1)sA\mu_w^{b-2}\big(1+\mu_r\big)\big) \Big)J^{-1}$$

$$K_{er} \equiv -aA\mu_w^{b-1} J^{-1}$$

The MRS is a linear function of the variances and the covariances of the random variables, which is an artifact of the way uncertainty was removed from the model.

Implicitly differentiating the first order condition we obtain

$$\frac{\partial s}{\partial \sigma_e^2} = \frac{aC_0}{U}$$

$$\frac{\partial s}{\partial \sigma_r^2} = \frac{C_1 \left((\delta_1 + 1)C_3 - C_2\right)}{U}$$

$$\frac{\partial s}{\partial \sigma_w^2} = \frac{C_4 + (\delta_1 + 1)C_5}{U}$$

$$\frac{\partial s}{\partial \rho_{ew}} = \frac{C_6 + (\delta_1 + 1)C_7}{U}$$

$$\frac{\partial s}{\partial \rho_{wr}} = \frac{C_8 + (\delta_1 + 1)C_9}{U}$$

$$\frac{\partial s}{\partial \rho_{er}} = \frac{C_{10} \left((\delta_1 + 1)C_{12} - C_{11}\right)}{U}$$

where

$$C_0 \equiv \frac{1}{2}(\delta_1 + 1)(A\mu_w^{b-1})^2 (v_1)^{-(\delta_1+1)} (v^e)^{\delta_1}$$

$$C_1 \equiv asA(\mu_w^{b-1})^2$$

$$C_2 \equiv \beta \, w_1^{1-b}$$

$$C_3 \equiv \frac{1}{2}sA(v_1)^{-(\delta_1+1)} (v^e)^{\delta_1}$$

$$C_4 \equiv \frac{1}{2}\beta \, \mu_w^{b-2} w_1^{b-1} (1-b)(1+\mu_r)(2aG_1 + (2-b)\mu_w^{-1})$$

$$C_5 \equiv \frac{1}{2}(aG_1^2 - G_2)(v_1)^{-(\delta_1+1)} (v^e)^{\delta_1}$$

$$C_6 \equiv a(1-b)(1+\mu_r)\beta \, A\mu_w^{2b-3} w_1^{1-b} \sigma_e \sigma_w$$

$$C_7 \equiv A\mu_w^{b-1}(aG_1 + (1-b)\mu_w^{-1}(v_1)^{-(\delta_1+1)} (v^e)^{\delta_1} \sigma_e \sigma_w$$

$$C_8 \equiv \beta \left(\frac{\mu_w}{w_1}\right)^{b-1} \left(a((1-b)sA\mu_w^{b-2}(1+\mu_r) - G_1) - (1-b)\mu_w^{-1}\right)\sigma_w \sigma_r$$

$$C_9 \equiv sA\mu_w^{b-1}\left((1-b)\mu_w^{-1} + aG_1\right)(v_1)^{-(\delta_1+1)} (v^e)^{\delta_1} \sigma_w \sigma_r$$

$$C_{10} \equiv aA(\mu_w^{b\text{-}1})^2 \sigma_e \sigma_r$$

$$C_{11} \equiv \beta \, w_1^{1\text{-}b}$$

$$C_{12} \equiv sA(v_1)^{-(\delta_1+1)} (v^e)^{\delta_1}$$

$$U \equiv -\frac{\partial F}{\partial s} J > 0 \quad \text{(by the SOSC)}$$

$$J \equiv 1/\left(1 + \frac{1}{2}\left(aA\mu_w^{b\text{-}1}\right)^2 \sigma_e^2 + \frac{1}{2}a\left(aG_1^2 - G_2\right)\sigma_w^2 + \frac{1}{2}\left(aA\mu_w^{b\text{-}1} s\right)^2 \sigma_r^2 \right.$$

$$+ aA\mu_w^{b\text{-}1}\left(aG_1 + (1\text{-}b)\mu_w^{-1}\right)\sigma_{ew} + asA\mu_w^{b\text{-}1}\left(aG_1 + (1\text{-}b)\mu_w^{-1}\right)\sigma_{wr}$$

$$\left. + \left(aA\mu_w^{b\text{-}1}\right)^2 s\, \sigma_{er} \right) > 0$$

and where the "U" term in each comparative statics expression is evaluated with the appropriate variance and covariance terms set equal to zero.

APPENDIX B: CHAPTER 4 DERIVATIONS

Here we will derive the results discussed in Chapter 4. The univariate results can be obtained by setting the unwanted variance and covariance terms equal to zero and replacing mean values with certainty values. As in appendix A, the arguments of the two indirect utility functions are dropped for notational ease.

STOCHASTIC INCOME

For the case of joint endowment, wage, and capital income uncertainty the reduced optimization problem is

$$\underset{s}{\text{Max}} \quad E(U(v_1,\tilde{v}_2)) = -\frac{1}{\delta_1}[(v_1)^{-\delta_1} + \beta\, E(\tilde{v}_2)^{-\delta_1}]$$

s.t.

$$v_1 = Aw_1^{b-1} y_1 = Aw_1^{b-1}(e_1 + w_1 - s)$$

$$\tilde{v}_2 = A\tilde{w}_2^{b-1} \tilde{y}_2 = A\tilde{w}_2^{b-1}(\tilde{e}_2 + \tilde{w}_2 + s(1+\tilde{r}))$$

$$\begin{bmatrix} \tilde{e}_2 \\ \tilde{w}_2 \\ (1+\tilde{r}) \end{bmatrix} \sim \left(\begin{bmatrix} \mu_e \\ \mu_w \\ (1+\mu_r) \end{bmatrix}, \begin{bmatrix} \sigma_e^2 & \rho_{ew}\sigma_e\sigma_w & \rho_{er}\sigma_e\sigma_r \\ \rho_{ew}\sigma_e\sigma_w & \sigma_w^2 & \rho_{wr}\sigma_w\sigma_r \\ \rho_{er}\sigma_e\sigma_r & \rho_{wr}\sigma_w\sigma_r & \sigma_r^2 \end{bmatrix} \right)$$

Taking the expectation of the second order multivariate Taylor approximation of

$(\tilde{v}_2)^{-\delta_1}$ around the point $(\mu_e, \mu_w, (1+\mu_r))$ we obtain

$$E(\tilde{v}_2)^{-\delta_1} \cong (\bar{v}_2)^{-\delta_1} + \frac{1}{2}\delta_1(\delta_1+1)(A\mu_w^{b-1})^2 (\bar{v}_2)^{-(\delta_1+2)}\sigma_e^2$$

$$+ \frac{1}{2}\delta_1\left((\delta_1+1)G_1^2(\bar{v}_2)^{-(\delta_1+2)} - G_2(\bar{v}_2)^{-(\delta_1+1)}\right)\sigma_w^2$$

$$+ \frac{1}{2}\delta_1(\delta_1+1)(A\mu_w^{b-1}s)^2(\bar{v}_2)^{-(\delta_1+2)}\sigma_r^2$$

$$+ \delta_1\left((\delta_1+1)(A\mu_w^{b-1})G_1(\bar{v}_2)^{-(\delta_1+2)} - (b-1)A\mu_w^{b-2}(\bar{v}_2)^{-(\delta_1+1)}\right)\sigma_{ew}$$

$$+ \delta_1\left((\delta_1+1)(A\mu_w^{b-1}s)G_1(\bar{v}_2)^{-(\delta_1+2)} - (b-1)A\mu_w^{b-2}s(\bar{v}_2)^{-(\delta_1+1)}\right)\sigma_{wr}$$

$$+ \delta_1(\delta_1+1)(A\mu_w^{b-1})^2(\bar{v}_2)^{-(\delta_1+2)}s\sigma_{er}$$

$$\equiv v_2^e$$

where

$$\bar{v}_2 \equiv A\mu_w^{b-1}(\mu_e+\mu_w+s(1+\mu_r))$$

$$G_1 \equiv A\mu_w^{b-2}((b-1)(\mu_e+s(1+\mu_r)) + b\mu_w)$$

$$G_2 \equiv (b-1)A\mu_w^{b-3}((b-2)(\mu_e+s(1+\mu_r)) + b\mu_w)$$

The reduced optimization problem can be written as

$$\underset{s}{\text{Max}} \quad E(U(v_1,\tilde{v}_2)) = -\frac{1}{\delta_1}[(v_1)^{-\delta_1} + \beta\, v_2^e]$$

s.t.

$$v_1 \equiv Aw_1^{b-1}(e_1+w_1-s)$$

$$v_2^e \equiv (\bar{v}_2)^{-\delta_1} + \frac{1}{2}\delta_1(\delta_1+1)(A\mu_w^{b-1})^2(\bar{v}_2)^{-(\delta_1+2)}\sigma_e^2$$

$$+ \frac{1}{2}\delta_1\left((\delta_1+1)G_1^2(\bar{v}_2)^{-(\delta_1+2)} - G_2(\bar{v}_2)^{-(\delta_1+1)}\right)\sigma_w^2$$

$$+ \frac{1}{2}\delta_1(\delta_1+1)(A\mu_w^{b-1}s)^2(\bar{v}_2)^{-(\delta_1+2)}\sigma_r^2$$

$$+ \delta_1\left((\delta_1+1)(A\mu_w^{b-1})G_1(\bar{v}_2)^{-(\delta_1+2)} - (b-1)A\mu_w^{b-2}(\bar{v}_2)^{-(\delta_1+1)}\right)\sigma_{ew}$$

$$+ \delta_1\left((\delta_1+1)(A\mu_w^{b-1}s)G_1(\bar{v}_2)^{-(\delta_1+2)} - (b-1)A\mu_w^{b-2}s(\bar{v}_2)^{-(\delta_1+1)}\right)\sigma_{wr}$$

$$+ \ \delta_1(\delta_1+1)(A\mu_w^{b-1})^2\,(\bar{v}_2)^{-(\delta_1+2)}\,s\sigma_{er}$$

After some simplification, and defining

$$\mu_y \equiv \mu_e + \mu_w + s(1+\mu_r)$$

the reduced form first order condition is

$$\left(\frac{v_1}{\bar{v}_2}\right)^{-(\delta_1+1)} = \beta\left(\frac{\mu_w}{w_1}\right)^{b-1}\left[(1+\mu_r) + K_e\sigma_e^2 + K_w\sigma_w^2 + K_r\sigma_r^2 + K_{ew}\sigma_{ew} + K_{wr}\sigma_{wr} + K_{er}\sigma_{er}\right]$$

where

$$K_e \equiv \frac{1}{2}(\delta_1+1)(\delta_1+2)\,\mu_y^{-2}\,(1+\mu_r)$$

$$K_w \equiv \frac{1}{2}(1+\mu_r)\Big(\ (b-1)(b-2)\,\mu_w^{-2} + (\delta_1+1)(\delta_1+2)G_1^2\,(\bar{v}_2)^{-2}$$

$$-\ (\delta_1+1)(\bar{v}_2)^{-1}(G_2-2(1-b)\,\mu_w^{-1}G_1)\ \Big)$$

$$K_r \equiv (\delta_1+1)\,\mu_y^{-1}\,s\left(\frac{1}{2}(\delta_1+2)\,\mu_y^{-1}\,s(1+\mu_r)-1\right)$$

$$K_{ew} \equiv (\delta_1+1)A\mu_w^{b-1}\,(\bar{v}_2)^{-2}\left(1+\mu_r\right)\!\left((\delta_1+2)\,\mu_w\,G_1-2(b-1)v_2\right)$$

$$K_{wr} \equiv (1-b)\,\mu_w^{-1}\left(2(\delta_1+1)\,\mu_y^{-1}\,s(1+\mu_r)-1\right) + (\delta_1+1)G_1\,(\bar{v}_2)^{-1}\!\left((\delta_1+2)\,\mu_y^{-1}\,s(1+\mu_r)-1\right)$$

$$K_{er} \equiv (\delta_1+1)\,\mu_y^{-1}\left((\delta_1+2)\,\mu_y^{-1}\,s(1+\mu_r)-1\right)$$

Implicitly differentiating the first order condition we obtain

$$\frac{\partial s}{\partial \sigma_e^2} = \frac{(\delta_1+1)D_1}{U}$$

$$\frac{\partial s}{\partial \sigma_r^2} = \frac{(\delta_1+1)D_2}{U}$$

$$\frac{\partial s}{\partial \sigma_w^2} = \frac{D_3+(\delta_1+1)D_4}{U}$$

$$\frac{\partial s}{\partial \rho_{ew}} = \frac{(\delta_1 + 1)D_5}{U}$$

$$\frac{\partial s}{\partial \rho_{wr}} = \frac{D_6 + (\delta_1 + 1)D_7}{U}$$

$$\frac{\partial s}{\partial \rho_{er}} = \frac{(\delta_1 + 1)D_8}{U}$$

where

$$D_1 \equiv \frac{1}{2}\beta\,(\delta_1 + 2)(1 + \mu_r)\,\mu_y^{-1}\left(\frac{\mu_w}{w_1}\right)^{b-1}$$

$$D_2 \equiv \beta\,s\mu_y^{-1}\left(\frac{\mu_w}{w_1}\right)^{b-1}\left(\frac{1}{2}(\delta_1 + 2)s(1 + \mu_r)\,\mu_y^{-1} - 1\right)$$

$$D_3 \equiv \frac{1}{2}\beta\,(b-1)(b-2)(1 + \mu_r)\,\mu_w^{b-3}\,w_1^{1-b}$$

$$D_4 \equiv \frac{1}{2}\beta\,(1 + \mu_r)\left(\frac{\mu_w}{w_1}\right)^{b-1}\left((\delta_1 + 2)(\overline{v}_2)^{-2}\,G_1^2 + (\overline{v}_2)^{-1}(2(1-b)\,\mu_w^{-1}\,G_1 - G_2)\right)$$

$$D_5 \equiv \beta\,(1+r)A\mu_w^{2b-3}\,w_1^{1-b}\,(\overline{v}_2)^{-2}\left((\delta_1 + 2)\,\mu_w\,G_1 + 2(1-b)\overline{v}_2\right)\sigma_e\sigma_w$$

$$D_6 \equiv \beta\,(1-b)\,\mu_w^{b-2}\,w_1^{1-b}\left(2(\delta_1 + 1)s(1 + \mu_r)\,\mu_y^{-1} - 1\right)\sigma_w\sigma_r$$

$$D_7 \equiv \beta\,G_1(\overline{v}_2)^{-1}\left(\frac{\mu_w}{w_1}\right)^{b-1}\left((\delta_1 + 2)s(1 + \mu_r)\,\mu_y^{-1} - 1\right)\sigma_w\sigma_r$$

$$D_8 \equiv \beta\,\mu_y^{-1}\left(\frac{w_2}{w_1}\right)^{b-1}\left((\delta_1 + 2)s(1 + \mu_r)\,\mu_y^{-1} - 1\right)\sigma_e\sigma_r$$

$$U \equiv -\frac{\partial F}{\partial s} > 0 \quad \text{(by the SOSC)}$$

and where the "U" term in each comparative statics expression is evaluated with the appropriate variance and covariance terms set equal to zero.